Chapter 2: Basic Neuron and Network Models.

Lecture 1

Statistical Models of Neuronal Activity

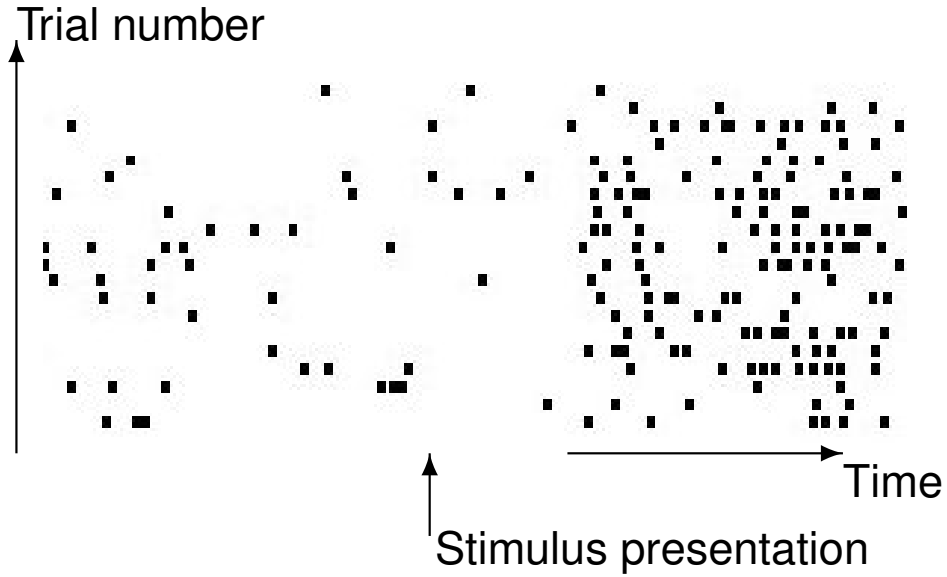
and Neural Coding

Neural coding

- External inputs are encoded by spike trains of neural populations that form the output of sensory systems (e.g., ganglion cells in the retina);
- These spike trains influence internal state of the central nervous system;
- Motor outputs are conveyed to target muscles in the form of spike trains;
- How is information about external inputs, internal state, and motor outputs encoded in spike trains of populations of neurons?

Neuronal variability

- Neuronal responses are variable from trial to trial;
- Statistical descriptions of neuronal activity are needed
- Characterize statistics of firing, conditioned by parameters describing sensory stimulus P(spike train $\mid$ stimulus)
- This conditional probability distribution can in principle reveal how much information about the stimulus is contained in spike trains

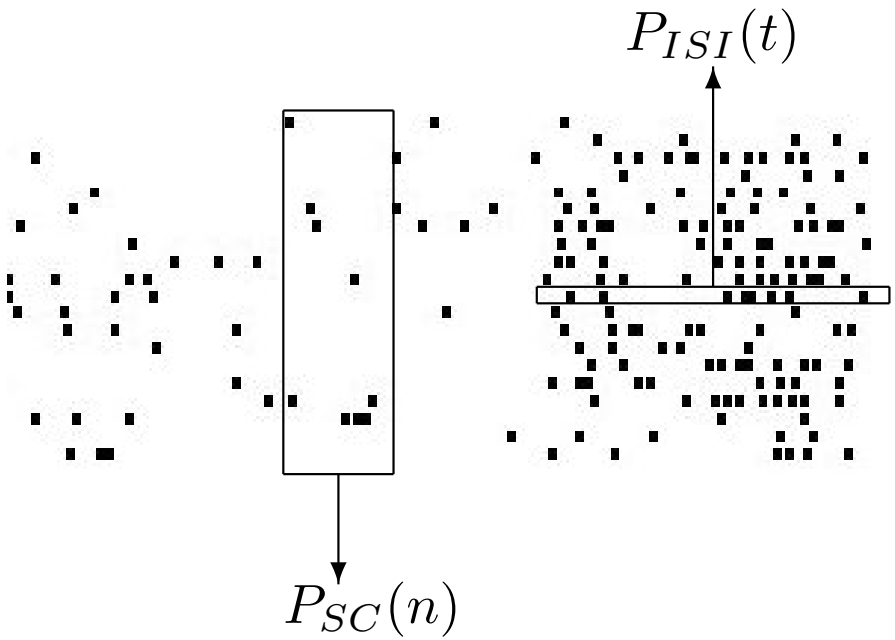


Spike trains

- Spike train $=$ list of spike times, $\{t_{ik}, i = 1, \ldots, N_T, k = 1, \ldots, N_i\}$
  - t_{ik}: time of k-th spike in trial number i
  - N_T: number of trials
  - N_i: number of spikes emitted during trial i
- Spikes are often described mathematically by delta functions at times t_{ik}
- For practical purposes, spike trains are often discretized in bins of width Δt
- For Δt short enough, spike trains then described by a string of T binary numbers $S_i(t), t = 1, \ldots, T,$
  - $S_i(t) = 1$: spike emitted in bin t in trial i
  - $S_i(t) = 0$: no spike emitted in bin t in trial i.

Spike train variability

- Variability within a trial:
  - Distribution of inter-spike intervals (ISI) $P_{ISI}(t)$
  - Mean ISI $=$ 1/(firing rate)
  - Standard deviation (SD) of ISI
  - Coefficient of variation (CV) $=$ SD/mean
- Variability across trials:
  - Define width of temporal window Δt
  - Compute distribution of spike counts in this window $P_{SC}(n)$
  - Compute mean and variance of spike counts
  - Fano factor (FF) $=$ variance/mean



Spike train as point processes: The Poisson process

- Point process: stochastic process whose realizations consist of a set of isolated points in time
- Simplest point process: the **Poisson process**
- Homogeneous **Poisson processes**:
  - Independence of successive ISIs
  - Exponential distribution of ISIs, $P(t) = \nu \exp(-\nu t)$
  - $\nu =$ spike rate/firing rate (in spikes per second)
  - Mean ISI $= 1/\nu$
  - CV $= 1$
  - FF $= 1$
  - Distribution of spike counts in an interval of duration T:

$$P(n) = \frac{(\nu T)^n}{n!} \exp(-\nu T)$$

- Inhomogeneous Poisson process:
  - Time-varying firing rate
  - Probability of spike emission in a given interval depends only on firing rate in that interval

Poisson processes vs. spike trains of real neurons

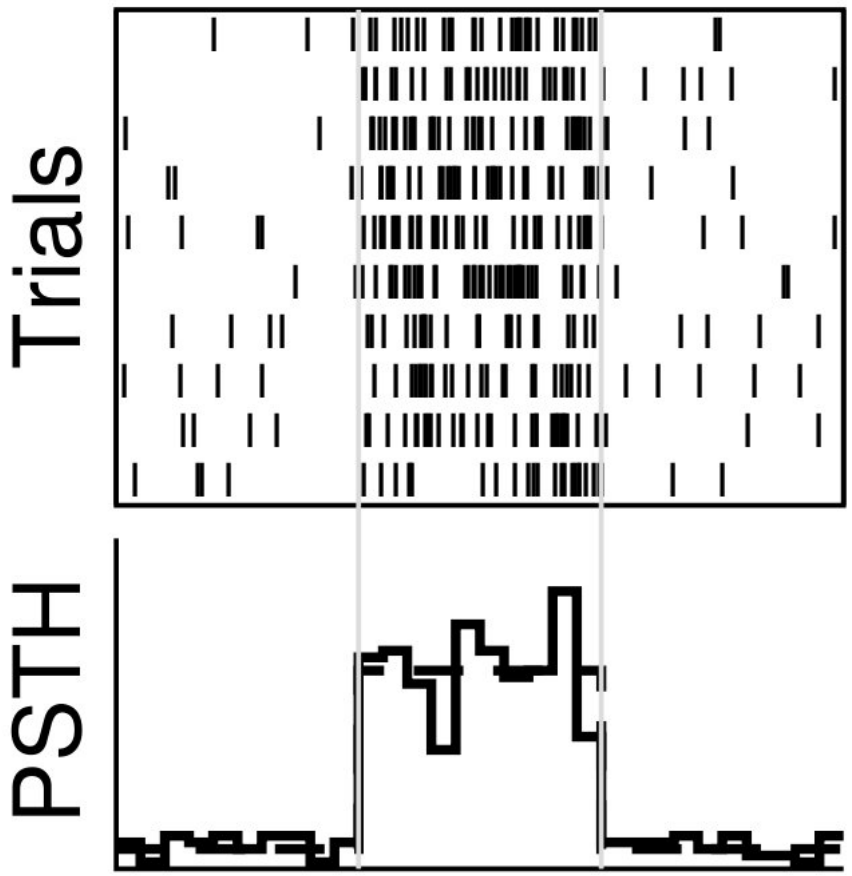
- Similarities with spike trains recorded in vivo:
  - Broad distributions of ISIs
  - CVs often close to one (particularly in cortex, ...)
  - Weak correlations between successive ISIs
- Differences with spike trains recorded in vivo:
  - ISIs of real neurons cannot be shorter than absolute refractory period
  - A large fraction of neurons have CVs significantly different from 1

Other point process models

- Poisson process with dead time
- Renewal process with gamma distributed ISIs
- Renewal process with inverse Gaussian distributed ISIs
- Spike trains generated as output of specific biophysical neuron models (e.g., integrate-and-fire) with noisy input

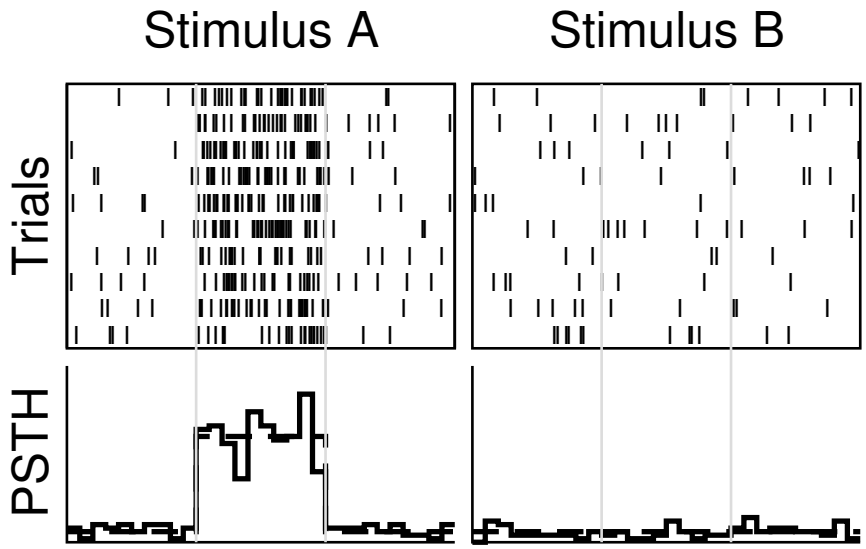
Averaging over trials: The poststimulus time histogram (PSTH)

- Averaging spike trains over trials $\rightarrow$ PSTH
- Divide time in bins
- PSTH $=$ (total number of spikes over all trials in each bin)/(number of trials times duration of bin)
- Gives an estimate of instantaneous firing rate vs. time
- Expected fluctuations decrease as $1/\sqrt{N_T}$ where N_T is the number of trials



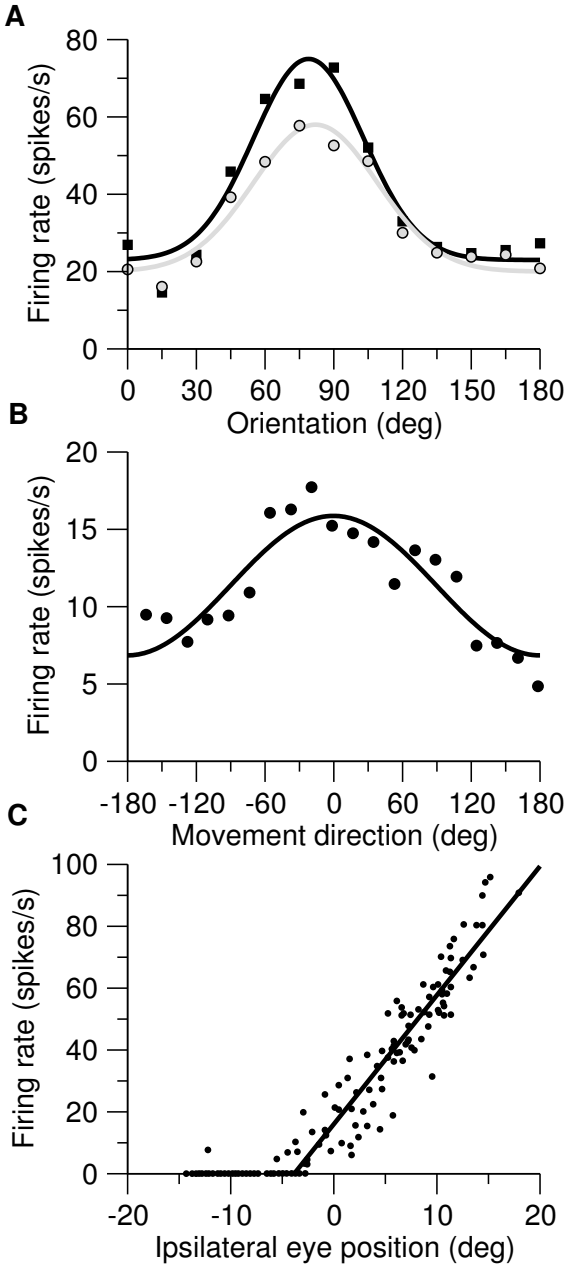
Rate coding

- In many cases, the firing rate depends on which stimulus is shown
- Spike count over a temporal window contains information about the stimulus
- This form of coding is sometimes referred to as **rate coding**



How firing rates depend on external stimuli: Tuning curves

- **Tuning curve**: how firing rate depends on a (typically continuous) parameter characterizing the stimulus
- Experimentally recorded tuning curves have been fitted by various functions:
  - Bell-shaped tuning curves:
    - Gaussian (e.g., firing rate vs. orientation in neurons in monkey MT (A), adapted from McAdams and Maunsell 1995)
    - Cosine (e.g., firing rate vs. arm movement direction in monkey M1 (B), adapted from Amirikian et al. 2000)
  - Monotonic tuning curves:
    - Threshold-linear (e.g., firing rate vs. eye position in goldfish oculomotor system, adapted from Aksay et al. 2000)

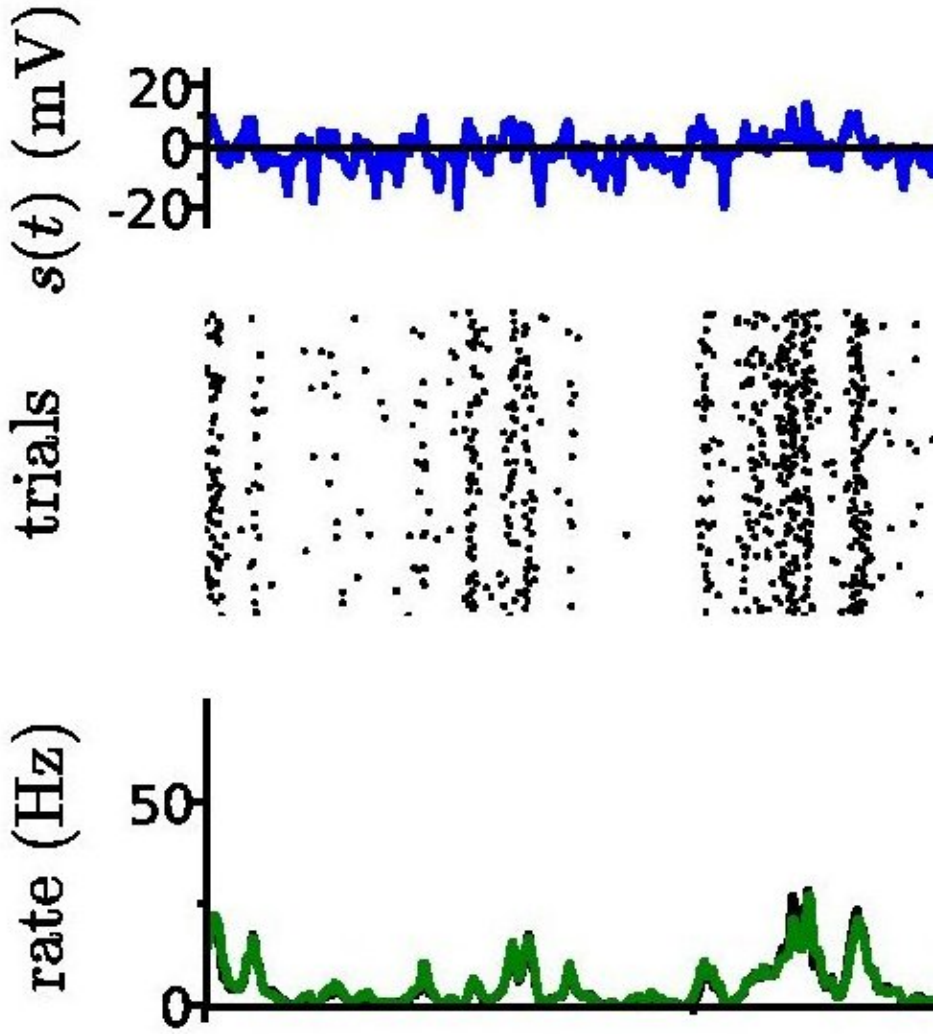


How are dynamic stimuli encoded by single neurons?

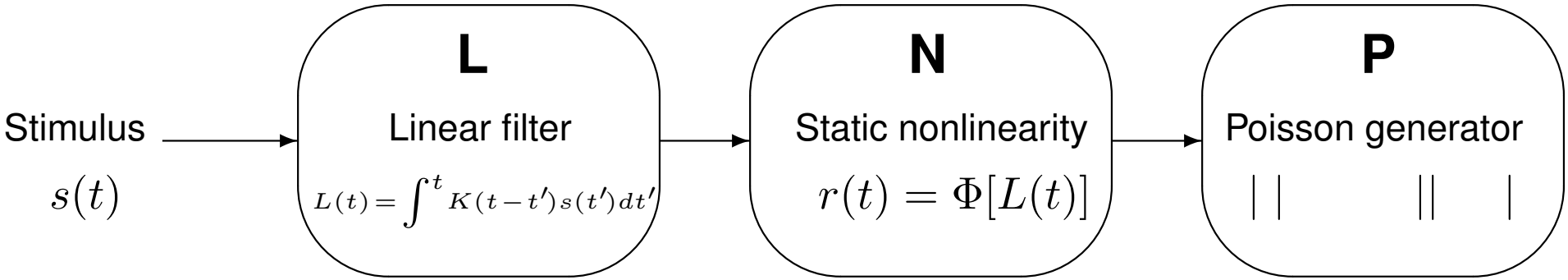
Time-dependent stimulus

$\Rightarrow$ Spike response to stimulus in different trials

$\Rightarrow$ Instantaneous firing rate (PSTH)



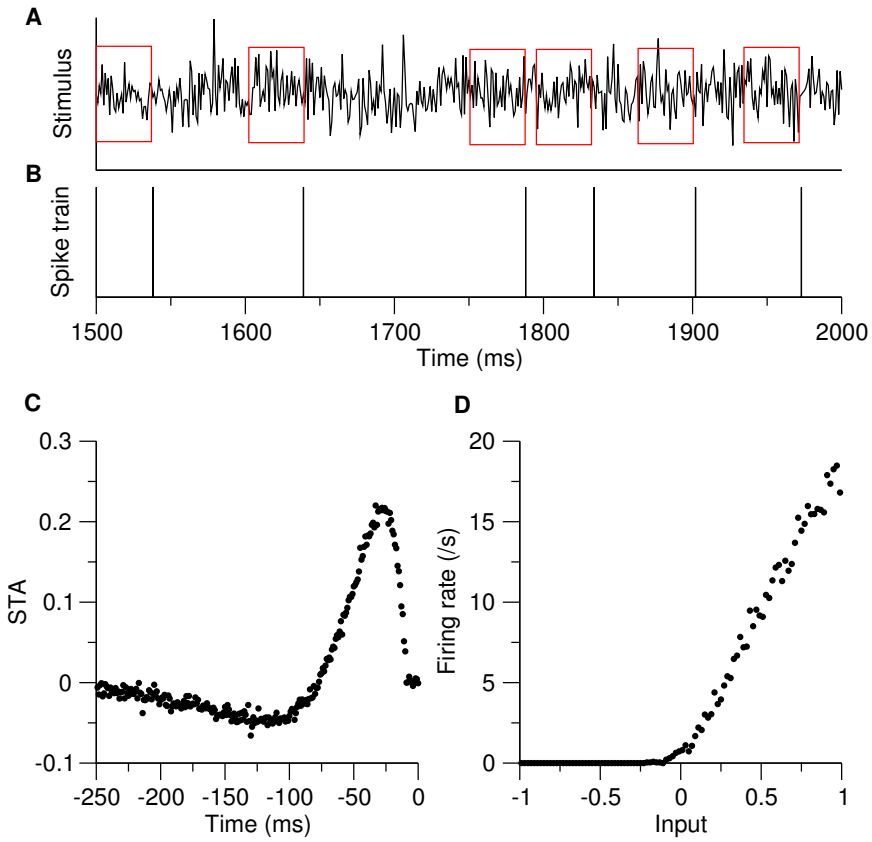
Linear-Nonlinear-Poisson (LNP) models



Fitting an LNP model to data

- First obtain the linear filter K using a **spike-trigger average** (STA) of the stimulus

- Then, obtain the static nonlinearity by computing the average rate $r(t)$ for each value of L



Receptive fields (RFs)

- Visual stimuli depend on space and time
- Linear filter depends on space and time $K(x, y, t)$ (spatiotemporal receptive field)
- Spatial receptive field: spatial area in which $K(x, y, t)$ is significantly different from zero
- Separable receptive fields:

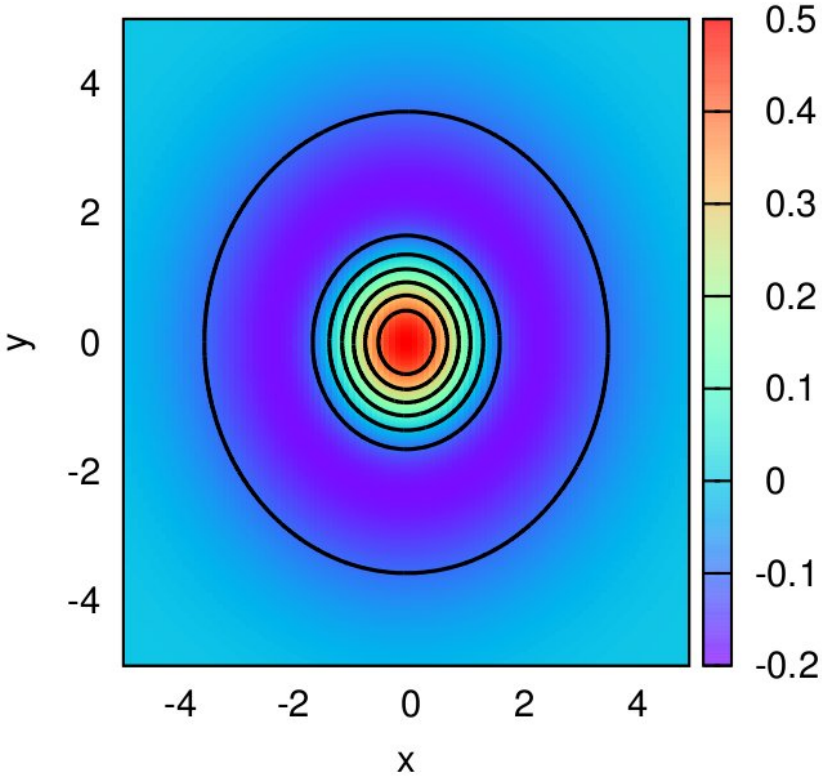
$$K(x, y, t) = K_s(x, y)K_t(t)$$

Receptive field examples - retina and thalamus

- In retina and thalamus, circularly symmetric RFs
- Modeled by difference of Gaussians, with ON (or OFF) center region, and OFF (or ON) surround region

$$K_s(x,y) = \pm\left(\frac{1}{2\pi\sigma_c^2}\exp\left(-\frac{x^2+y^2}{2\sigma_c^2}\right) - \frac{B}{2\pi\sigma_s^2}\exp\left(-\frac{x^2+y^2}{2\sigma_s^2}\right)\right)$$

  - $B =$ balance between center and surround
  - $\sigma_c =$ width of center
  - $\sigma_s =$ width of surround



Receptive field examples - V1

- In V1, RFs are no longer circularly symmetric
- Modeled by Gabor function

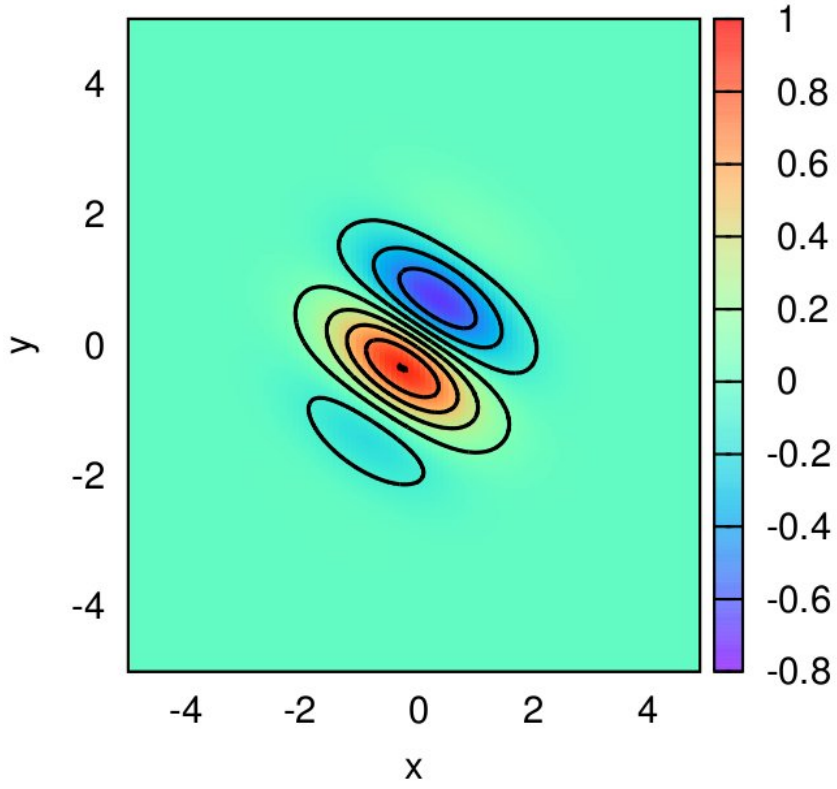
$$g_s(x, y) = \exp\left(-\frac{x'^2 + \gamma^2 y'^2}{2\sigma^2}\right) \cos\left(2\pi \frac{x'}{\lambda} + \psi\right)$$

$$x' = x\cos(\theta) + y\sin(\theta)$$

$$y' = -x\sin(\theta) + y\cos(\theta)$$

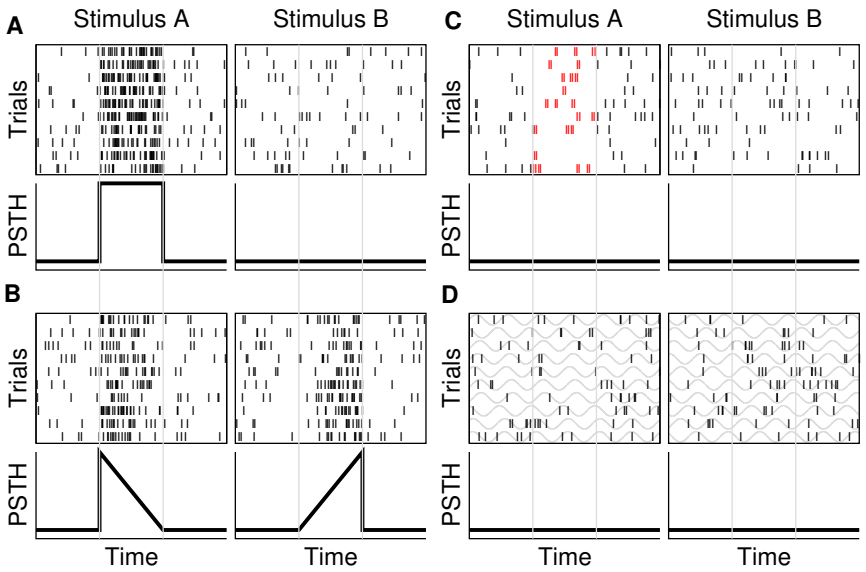
  - $\lambda =$ wavelength
  - $\theta =$ orientation
  - $\psi =$ phase offset
  - $\sigma =$ width of Gaussian envelope
  - $\gamma =$ spatial aspect ratio

- Leads to orientation selectivity



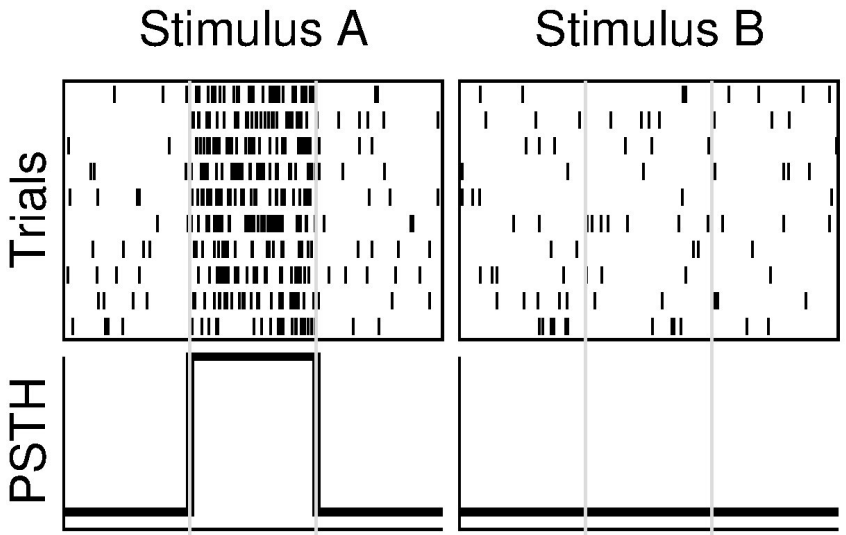
Single neuron coding: Beyond firing rates

- Models described so far: instantaneous firing rate is the only information-carrying quantity
- Alternate scenarios: spike correlations (spike pattern code); phase of spikes with respect to an oscillatory variable



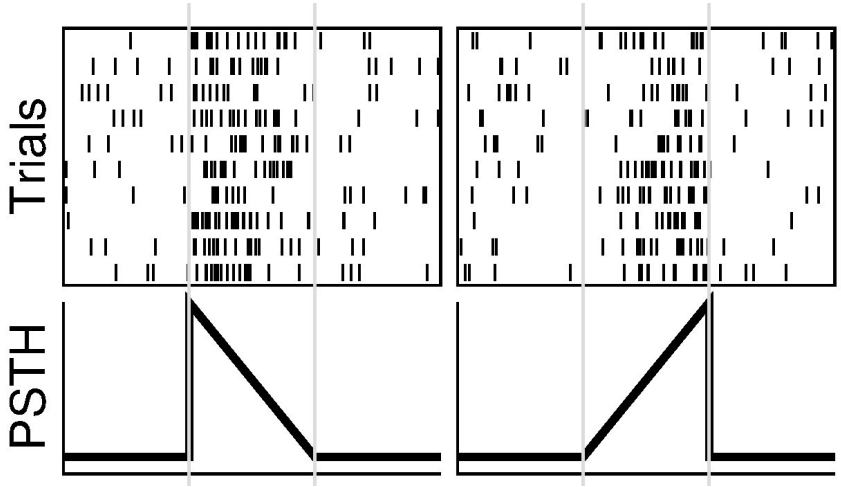
Static rate (spike count) code

- Information contained in the average spike rate (spike count over a time window)



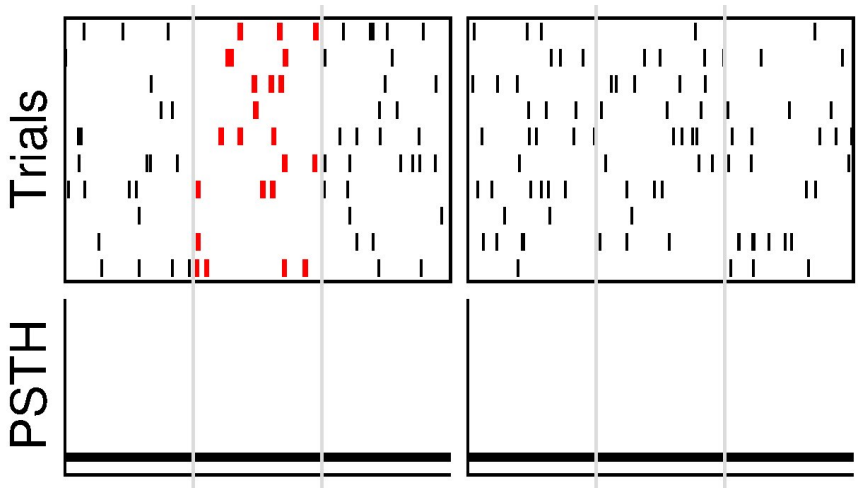
Time-varying rate code

- Information contained in the temporal dynamics of the instantaneous firing rate, but not in its mean



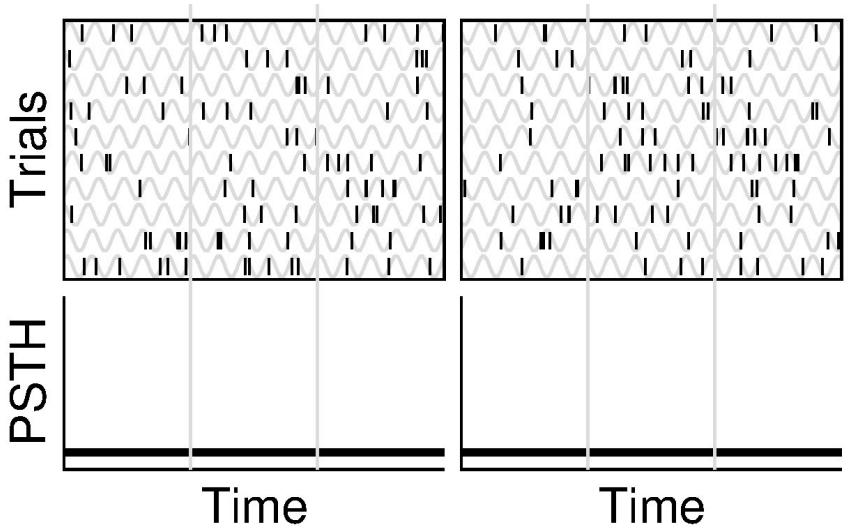
Correlation (spike pattern) code

- Information could be carried in higher order statistics of spike trains
- An example is a ”spike pattern” code - a stimulus is represented by a specific spike pattern (in the example on the right, a burst of two spikes)



Phase of firing code

- Brain networks often exhibit oscillatory activity
- Information can be contained in the phase of firing with respect to the on-going oscillation

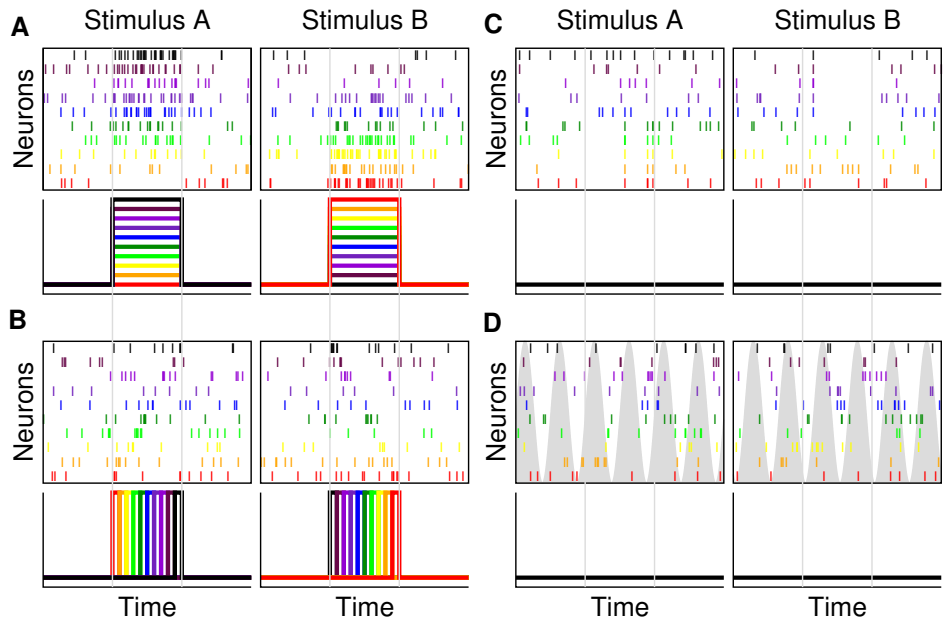


Population coding

- To understand coding by populations of neurons, we need statistical descriptions of population activity and how it depends on external stimuli
- Simplest model: independent neurons (no correlations)
- In some cases, independent model provides a poor description of population activity
- Models for correlated activity:
  - Maximal entropy models with second-order interactions
  - Dichotomized Gaussian models
  - etc.

Population coding scenarios

- Information carried by average firing rates of neurons (A)
- Information carried by temporal dynamics of firing rate (B)
- Correlation code (C)
- Phase of firing code (D)
- Several encoding schemes could be present simultaneously (multiplexing)



Which coding strategies are used by the brain?

- In the vast majority of brain areas investigated so far, spike counts carry information about external stimuli
- In many cases, temporal structure of instantaneous firing rate carries additional significant information
- In a few cases, higher order correlations and/or phase of firing have been shown to carry additional significant information
- How much information is carried by these different ”codes”?

  Can be quantified using tools from information theory

Quantifying information

- Stimulus $\theta \Rightarrow$ Neural response $\vec{r}$, whose statistics is given by $P(\vec{r}|\theta)$
- Shannon's mutual information

$$I = \int d\theta \rho(\theta) \int d\vec{r} P(\vec{r}|\theta) \log_2 \left(\frac{P(\vec{r}|\theta)}{P(\vec{r})} \right)$$

  - Quantifies how much the uncertainty about the stimulus is reduced by the neural response

- Fisher information

$$J(\theta) = \int d\vec{r} P(\vec{r}|\theta) \left(\frac{\partial}{\partial \theta} \log P(\vec{r}|\theta) \right)^2$$

  - Provides a lower bound on the variance of the error of any estimator of the stimulus
  - For N independent neurons, grows linearly with N (error decreases as $1/\sqrt{N}$)

Chapter 2: Basic Neuron and Network Models.

Lecture 2

Rate Models

Firing rate models

- Describe population activity not in terms of individual neuron variables, but rather in terms of average activity
- Useful to describe phenomena at the macroscopic or mesoscopic levels
- Population activity described by ordinary differential equations (ODEs)
- Known under various names: neural mass model, firing rate model, rate model, Wilson-Cowan model, neural field model

Simplest rate model

- Population activity $r(t)$ described by

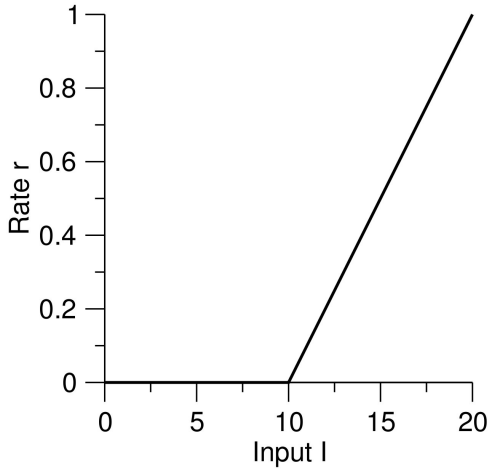
$$\tau \frac{dr(t)}{dt} = -r(t) + \Phi\left(I(t) + Jr(t)\right)$$

- τ: time constant of firing rate dynamics
- $\Phi(.)$: static transfer function (f-I curve)
- $I(t)$: external input
- J: strength of synaptic connections within the population
  - $J > 0$: excitatory network
  - $J < 0$: inhibitory network

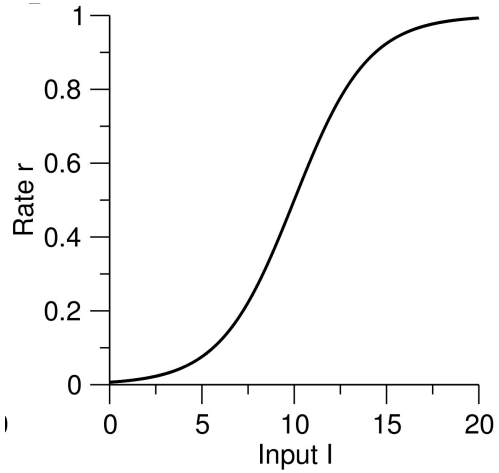
Amari 1972; Wilson and Cowan 1972

The transfer function Φ

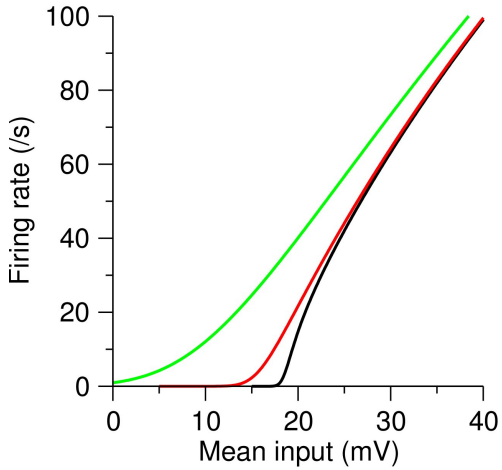
Threshold linear $\Phi(x) = [x - T]_+$



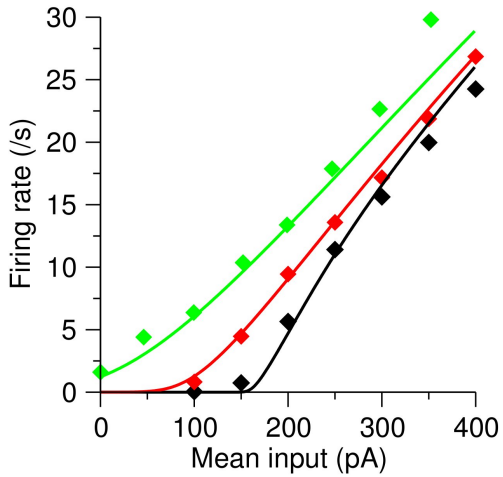
Sigmoidal $\Phi(x) = 1/(1 + \exp(-\beta(x - T)))$



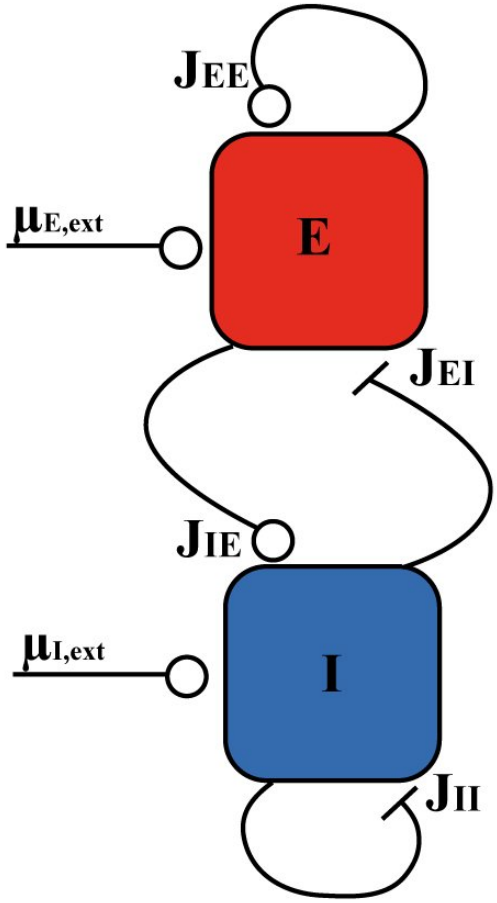
f-I curve of a specific spiking neuron model



f-I curve of a real neuron (Rauch et al. 2003)



Rate models for local populations of neurons



- n subpopulations described by their average firing rate r_i, $i = 1, \ldots, n$

$$\tau_i \frac{dr_i}{dt} = -r_i + \Phi_i \left(I_i + \sum_j J_{ij} r_j \right)$$

- Example: E-I network (Wilson and Cowan 1972):

$$\begin{aligned} \tau_E \frac{dr_E}{dt} &= -r_E + \Phi_E \left(I_{EX} + J_{EE} r_E - J_{EI} r_I \right) \\ \tau_I \frac{dr_I}{dt} &= -r_I + \Phi_I \left(I_{IX} + J_{IE} r_E - J_{II} r_I \right) \end{aligned}$$

Analysis of rate models

$$\tau \frac{dr}{dt} = -r + \Phi\left(I + \mathbf{J}r\right)$$

- Solve equations for fixed point(s):

$$r_0 = \Phi(I + \mathbf{J}r_0)$$

- Check linear stability of fixed point(s):
  - A small perturbation δr around the fixed point obeys the linearized dynamics

$$\dot{\delta} r = \frac{\left(-1 + \Phi' \mathbf{J}\right)}{\tau} \delta r$$

  - Compute eigenvalues λ of the Jacobian matrix $\left(-1 + \Phi' \mathbf{J}\right)$
  - Fixed point stable if all eigenvalues have negative real parts
  - "Rate" instability (saddle node bifurcation) when $\lambda = 0$
  - Oscillatory instability (Hopf bifurcation) when $\lambda = \pm i\omega$ and $\omega \neq 0$

Simplest case: One population, linear transfer function

$$\tau \frac{dr}{dt} = -r + (I + Jr)$$

- Unstable if $J > 1$ (rate instability)
- Perfect integrator if $J = 1$:

$$r(t) = \frac{1}{\tau} \int^{t} I(t')dt'$$

- Stable if $J < 1$:

$$\frac{\tau}{(1-J)} \frac{dr}{dt} = \frac{I}{(1-J)}$$

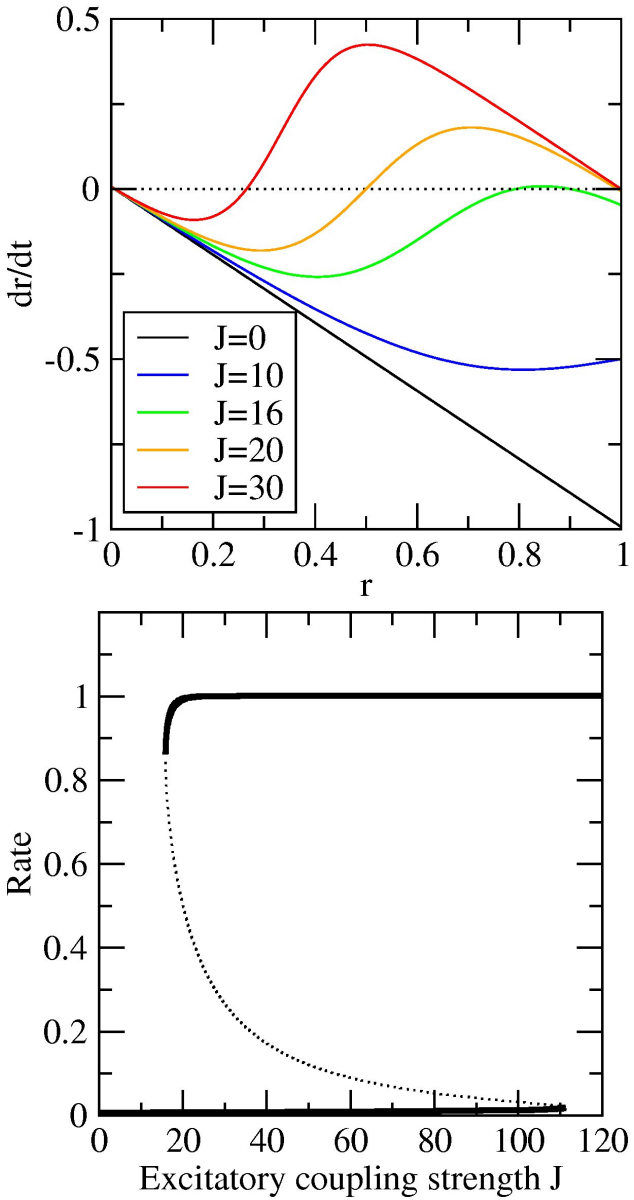
  - Excitatory network ($0 < J < 1$): amplification of inputs, slow response
  - Inhibitory network ($J < 0$): attenuation of inputs, fast response

Excitatory network: bistability

- Excitatory rate model:

$$\tau \frac{dr}{dt} = -r + \Phi\left(I + Jr\right)$$

- Fixed points and their stability can be obtained by plotting $\frac{dr}{dt}$ as a function of r
- For suitable I and Φ, there exists a range of J for which three solutions exist
- Low and high rate solutions are stable; intermediate rate solution unstable (gives the boundary of basins of attraction of both states)
- Bistable system keeps a memory of its initial condition



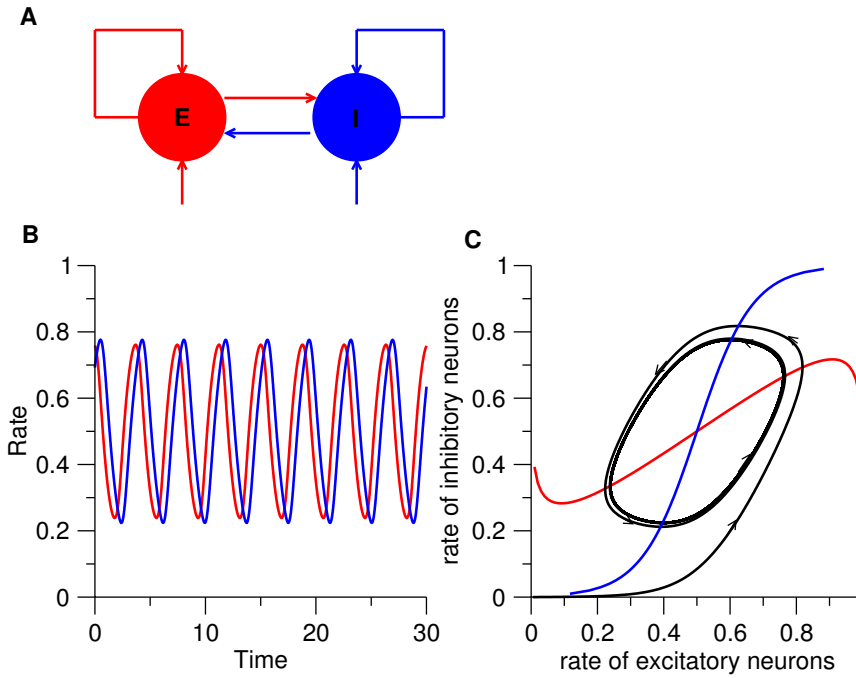
E-I network: oscillations

- Excitatory-inhibitory network:

$$\tau_E \frac{dr_E}{dt} = -r_E + \Phi_E \left(I_{EX} + J_{EE} r_E - J_{EI} r_I \right)$$

$$\tau_I \frac{dr_I}{dt} = -r_I + \Phi_I \left(I_{IX} + J_{IE} r_E - J_{II} r_I \right)$$

- Canonical model for local cortical networks
- Dynamics of two-variable models can be analyzed using phase plane analysis (see chapter 3)
- Can produce oscillations, provided both $J_{EI} J_{IE}$ and J_{EE} are sufficiently large

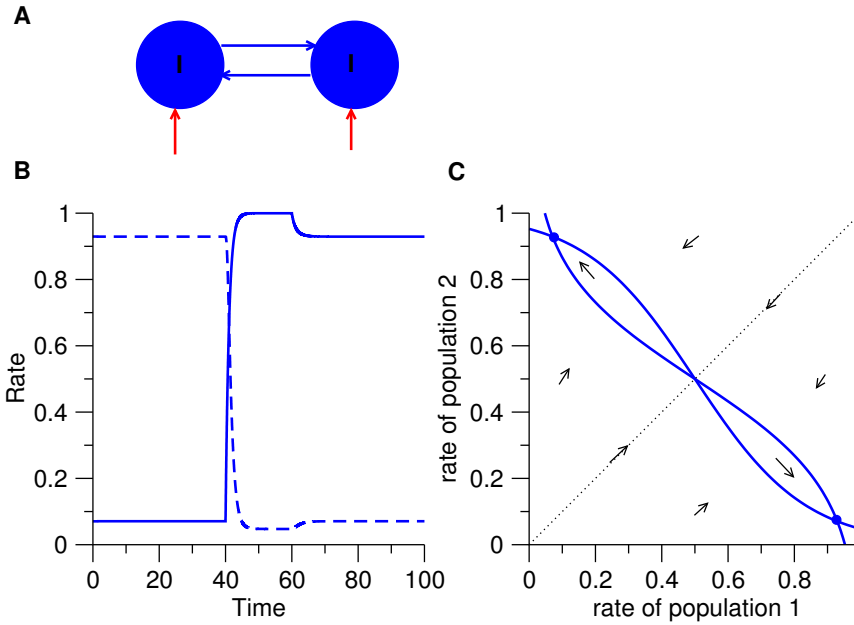


Multiple inhibitory populations - winner-take-all

- Simplest case: two mutually inhibitory populations

$$\tau \frac{dr_1}{dt} = -r_1 + \Phi(I_X - Jr_2)$$
$$\tau \frac{dr_2}{dt} = -r_2 + \Phi(I_X - Jr_1)$$

- For strong enough external inputs, and J, can become bistable
- In both stable states, one population has a high rate, the other a low rate
- "Winner-take-all" behavior



Spatially extended rate model (”neural field” model)

- Spatially extended rate model:

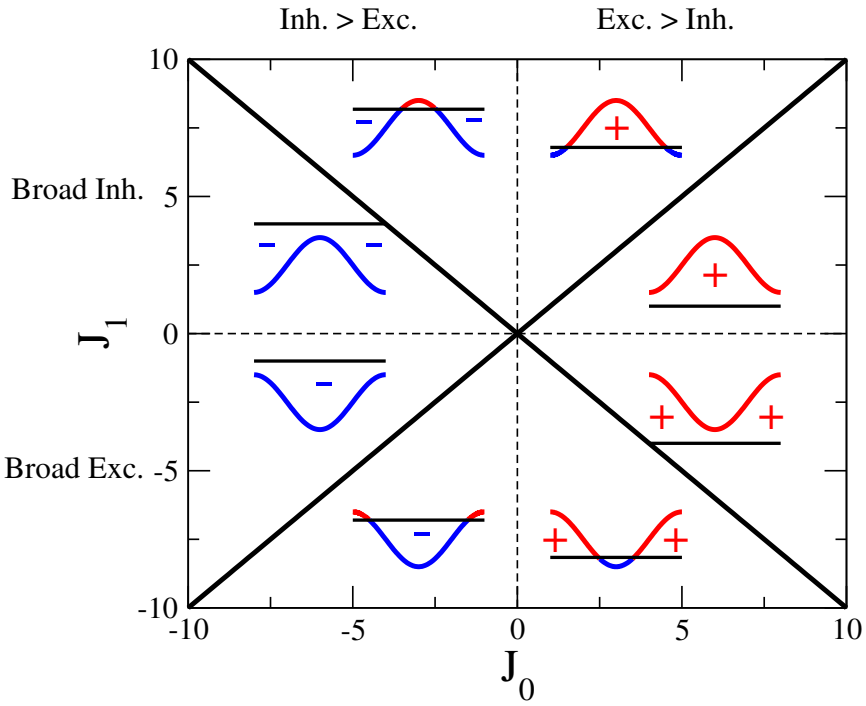
$$\tau \dot{r}(x,t) = -r(x,t) + \Phi\left(I(x,t) + \int dy J(|x-y|) r(y,t)\right)$$

- x can refer to spatial location, or to preferred stimulus of a population of neurons
- $r(x,t)$: firing rate of neurons at location x at time t
- $I(x,t)$: external input
- $J(|x-y|)$: weight of synaptic connections (”synaptic footprint”) between neurons at locations x and y (or with preferred stimuli x and y)

The ring model

- 1-D space with ring topology (orientation space): $x \in [-\pi, \pi]$
- Threshold-linear transfer function, $\Phi(I) = I$ if $I > 0$ and $\Phi(I) = 0$ otherwise
- Synaptic footprint:

$$J(|x - y|) = J_0 + J_1 \cos(x - y)$$



- This model, or variants of it, have been proposed as models of V1 (orientation selectivity), prefrontal cortex (spatial selectivity), head direction cells, place cells, grid cells, etc.

Ben-Yishai et al. 1995; Hansel and Sompolinsky 1998

Analysis of the model

Thanks to the simplified transfer function and footprint, the dynamics can be written in terms of three order parameters r_0 (**average activity**), r_1 (**spatial modulation of the activity**), and ψ (**location of the peak activity**):

$$\begin{aligned} r_0(t) &= \int \frac{dx}{2\pi} r(x,t) dx \\ r_1(t) &= \int \frac{dx}{2\pi} r(x,t) \cos(x - \psi(t)) dx \\ 0 &= \int \frac{dx}{2\pi} r(x,t) \sin(x - \psi(t)) dx \end{aligned}$$

These parameters evolve in time according to

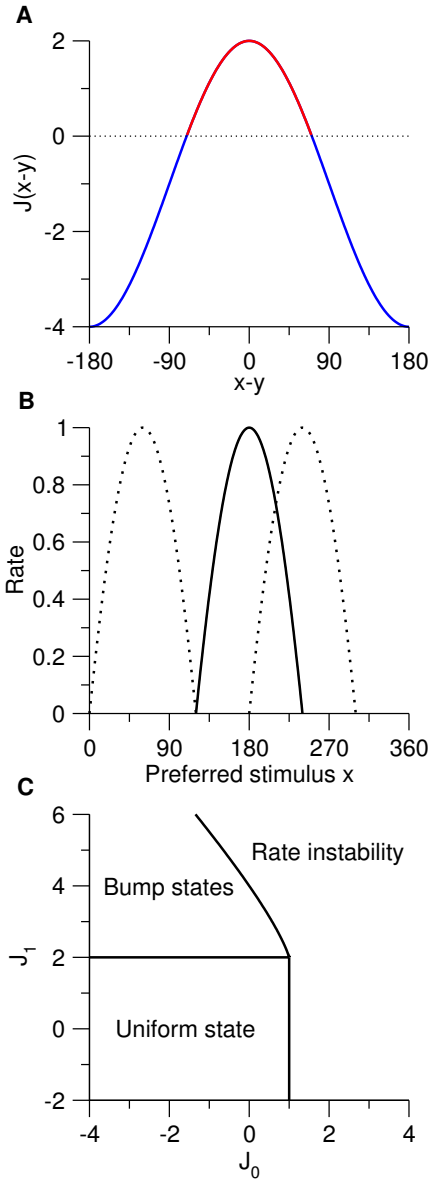
$$\begin{aligned} \dot{r_0}(t) &= -r_0(t) + \int \frac{dx}{2\pi} I(x) \\ \dot{r_1}(t) &= -r_1(t) + \int \frac{dx}{2\pi} \cos(x - \psi(t)) I(x) \\ \dot{\psi}(t) r_1(t) &= \int \frac{dx}{2\pi} \sin(x - \psi(t)) I(x) \\ I(x) &= \left[I^{ext} + J_0 r_0(t) + J_1 \cos(x - \psi(t)) r_1(t) \right]_+ \end{aligned}$$

Phase diagram of the model

- Stationary uniform state characterized by

$$r_0(t) = R_0, \quad r_1 = \psi = 0$$

- Stability analysis of stationary uniform state yields two types of instabilities:
  - Rate instability: $J_0 = 1$
  - Turing instability: $J_1 = 2$, leading to a ”bump state”
- Bump states have been proposed to account for spatial working memory in prefrontal cortex, and to account for properties of head direction cells, place cells, and grid cells



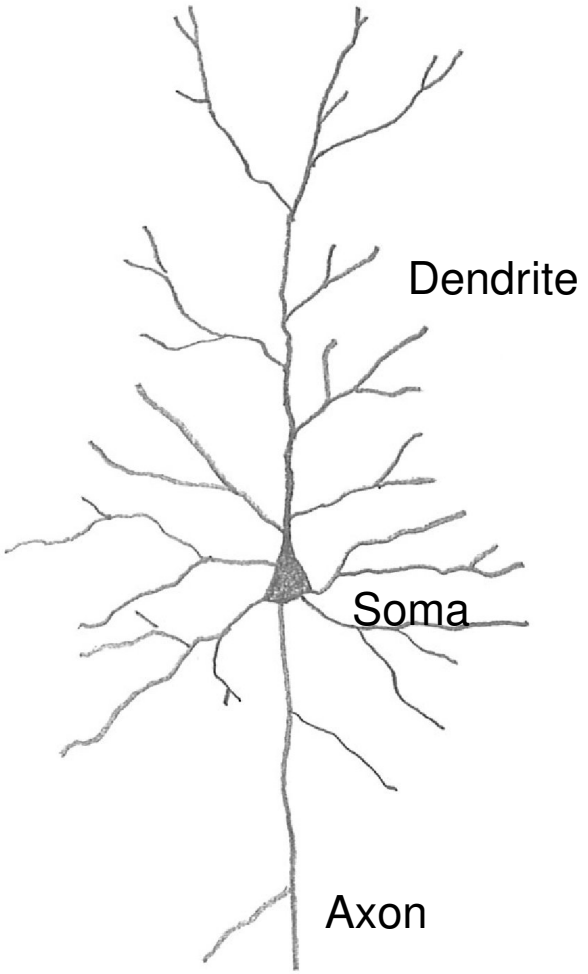
Chapter 2: Basic Neuron and Network Models.

Lecture 3

Single Neuron Models

Neurons

- Traditionally viewed as the basic computational units of the nervous system (”neuron doctrine”)
- Anatomically distinct cells with highly branched processes emerging from soma (dendrites and axons)
- Dendrites $=$ input, where vast majority of synaptic contacts are made
- In mammalian CNS, each neuron has typically thousands of synapses
- Axons $=$ output, propagate action potentials to postsynaptic neurons
- Neurons have been modeled at many different levels of complexity, from binary neurons to models described by thousands of ODEs



Hodgkin-Huxley (HH) model

- Describes the dynamics of the membrane potential, and variables describing voltage-gated ionic currents
- Introduced by Hodgkin and Huxley in their pioneering 1952 papers
- Hodgkin and Huxley obtained the model from fitting electrophysiological recordings from the squid giant axon
- HH formalism has been used to describe many more ionic currents than the two originally described by Hodgkin and Huxley, in many different cell types

Hodgkin-Huxley model - current balance equation

- Dynamics of the membrane potential obeys current balance equation

$$C\frac{dV}{dt} = -I_L(V) - I_{Na}(V) - I_K(V)$$

  in which four types of currents appear:

- The capacitive current $C\frac{dV}{dt}$ due to electrical charges on the membrane
- The leak current $I_L(V) = g_L(V - V_L)$ due to passive flow of ions through the membrane
- The sodium current $I_{Na}(V) = g_{Na}(V)(V - V_{Na})$ due to voltage-dependent opening of sodium channels inserted in the membrane
- The potassium current $I_K(V) = g_K(V)(V - V_K)$ due to voltage-dependent opening of potassium channels inserted in the membrane

Hodgkin-Huxley formalism - voltage-gated currents

- **Fast sodium current**

$$
\begin{aligned}
I_{Na}(V) &= \bar{g}_{Na} m^3 h (V - V_{Na}) \\
\tau_m(V) \frac{dm}{dt} &= -m + m_\infty(V) \\
\tau_h(V) \frac{dh}{dt} &= -h + h_\infty(V)
\end{aligned}
$$

  Provides positive feedback on the voltage:

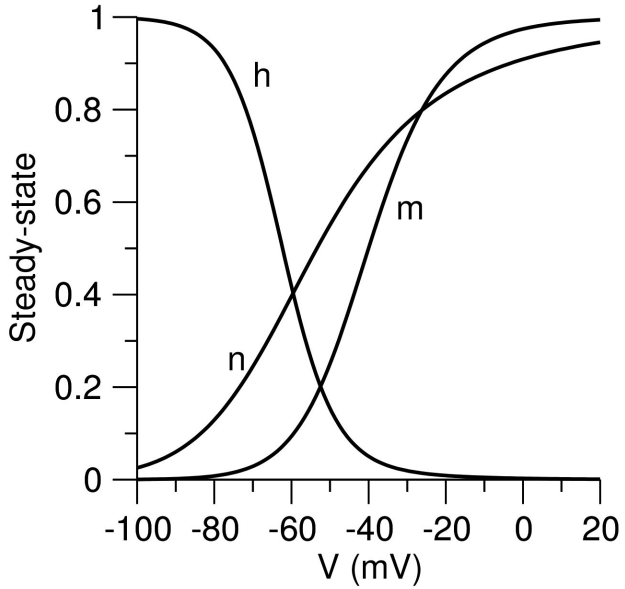
  - When V increases, m increases (Na channels open)
  - When Na channels open, Na ions enter the cell, leading to further increase in V
  - h decreases more slowly, eventually closing the channels

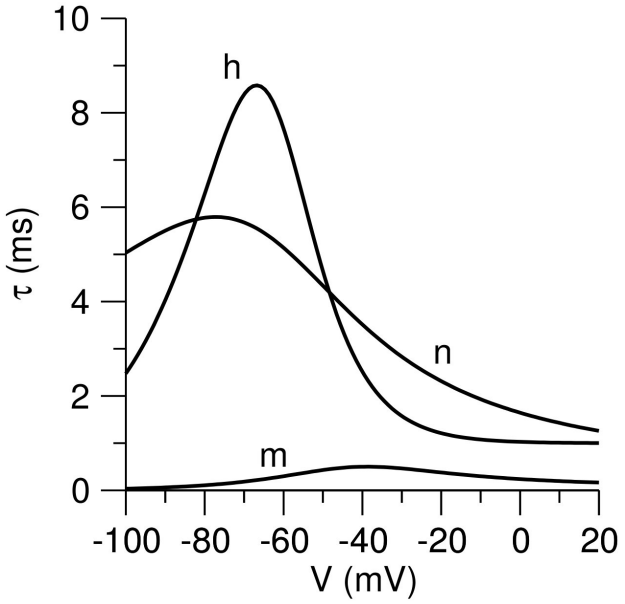
- **Delayed-rectifier potassium current**

$$
\begin{aligned}
I_K(V) &= \bar{g}_K n^4 (V - V_K) \\
\tau_n(V) \frac{dn}{dt} &= -n - n_\infty(V)
\end{aligned}
$$

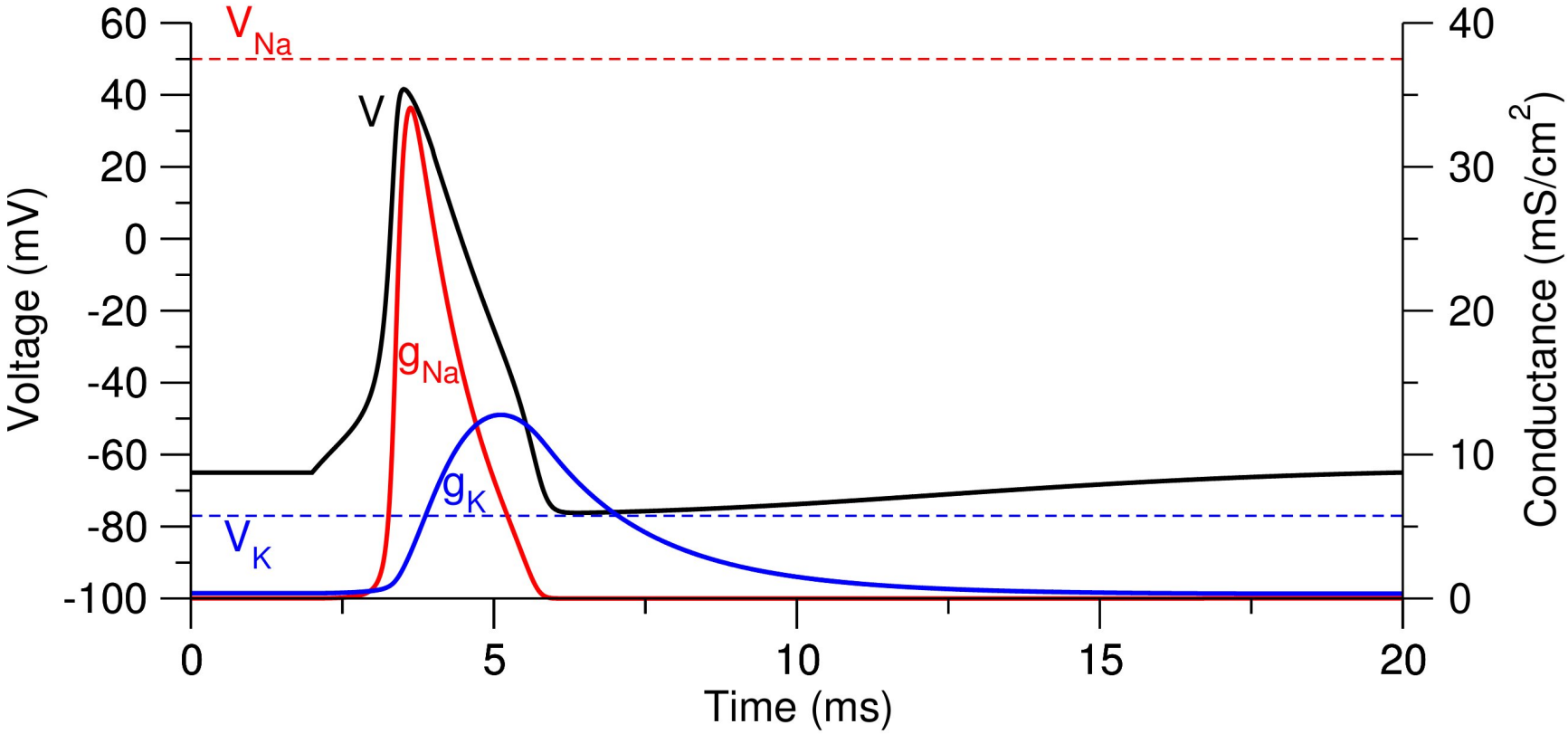
  Provides negative feedback on the voltage:

  - When V increases, n increases (K channels open)
  - When K channels open, K ions exit the cell, leading to a decrease in V





Putting everything together - the action potential in the HH model



Hodgkin-Huxley model - outlook

- Hodgkin-Huxley model has been highly influential
  - Allowed to understand mechanisms of action potential generation and propagation
  - Formalism used to describe quantitatively many other ionic currents
- Its complexity makes it difficult to analyze mathematically, as well as computationally expensive
- What about simpler models?

From HH to two variable models

- Variable m is much faster than all other variables:
  $\Rightarrow m(t) = m_\infty(V)$
- Dynamics of n and $1-h$ are similar:
  $\Rightarrow h = 1 - n$
- Gives a 2-D model:

$$\begin{aligned} C\frac{dV}{dt} &= -g_L(V - V_L) - \bar{g}_{Na}(V - V_{Na})m_\infty^3(V)(1-n) - \bar{g}_K n^4(V - V_K) \\ \tau_n(V)\frac{dn}{dt} &= -n + n_\infty(V) \end{aligned}$$

- Closely related models: Morris-Lecar; FitzHugh-Nagumo
- Can be analyzed using phase plane analysis (see chapter 3)

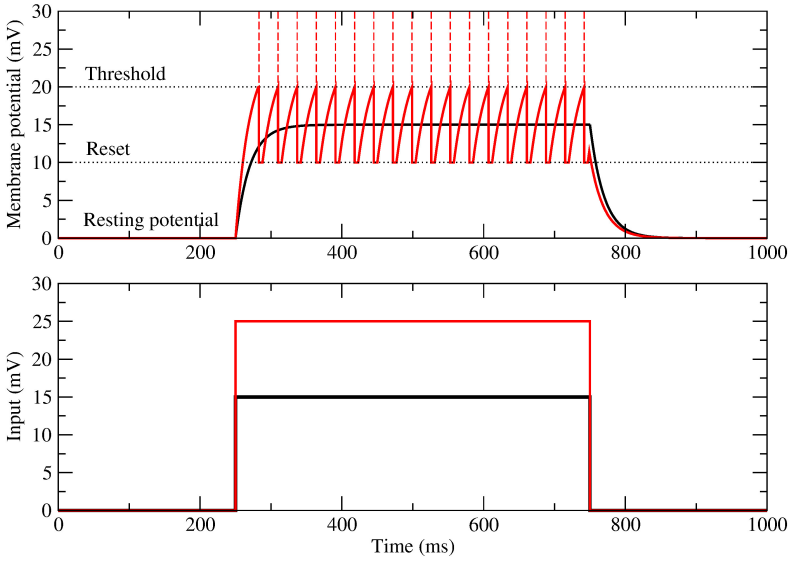
Leaky integrate-and-fire model

- Subthreshold dynamics ($V < V_T$) keep only capacitive and leak currents (Lapicque 1907):

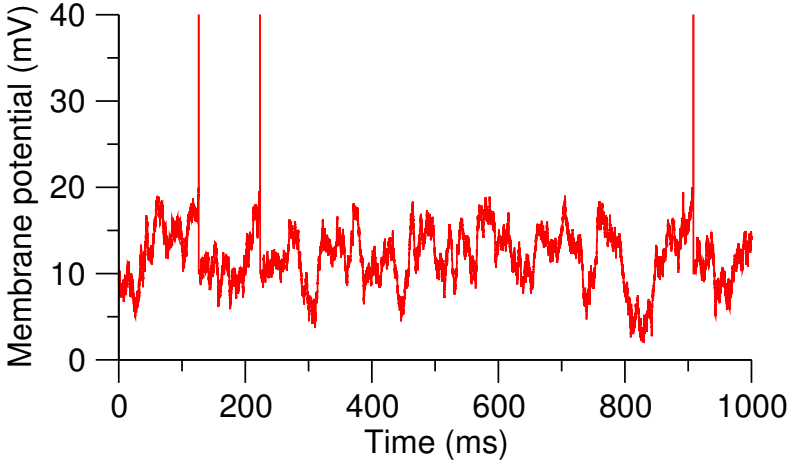
$$C\frac{dV}{dt} = -g_L(V - V_L) + I_{syn}(t)$$
$$\tau_m\frac{dV}{dt} = -V + \tilde{I}_{syn}(t)$$

- Spike emitted when $V = V_t$
- Then voltage reset to $V = V_r$
- (Optional) absolute refractory period of duration τ_{rp}
- V_t, V_r, and τ_{rp} replace Na and K currents in HH model

- Response to constant inputs



- Response to stochastic inputs



LIF: Pros and cons

- Pros
  - Computationally cheap - it is easy to simulate networks of tens of thousands of LIFs on a workstation)
  - Analytically tractable, in the presence of constant as well as stochastic inputs
  - Simple variants of LIF able to reproduce real neuron static transfer functions
- Cons
  - Cannot reproduce many features observed in real neurons (subthreshold resonance, bursting, firing rate adaptation, etc.)
- Simple extensions of the LIF model: try to get the best of both worlds (simplicity, ability to capture diversity of behaviors of real neurons)

Nonlinear integrate-and-fire models

- Spike generation dynamics captured by a nonlinear function of voltage

$$\tau_m \frac{dV}{dt} = -(V - V_L) + \psi(V) + I_{syn}(t)$$

- When ψ is supralinear, voltage diverges to infinity in finite time whenever the synaptic inputs exceed some threshold
- Time of divergence defines spike time in such models
- Popular choices of ψ:
  - **Quadratic integrate-and-fire (QIF)**

$$\psi(V) = \frac{1}{2\Delta_T}(V - V_T)^2 + (V - V_L) - I_T$$

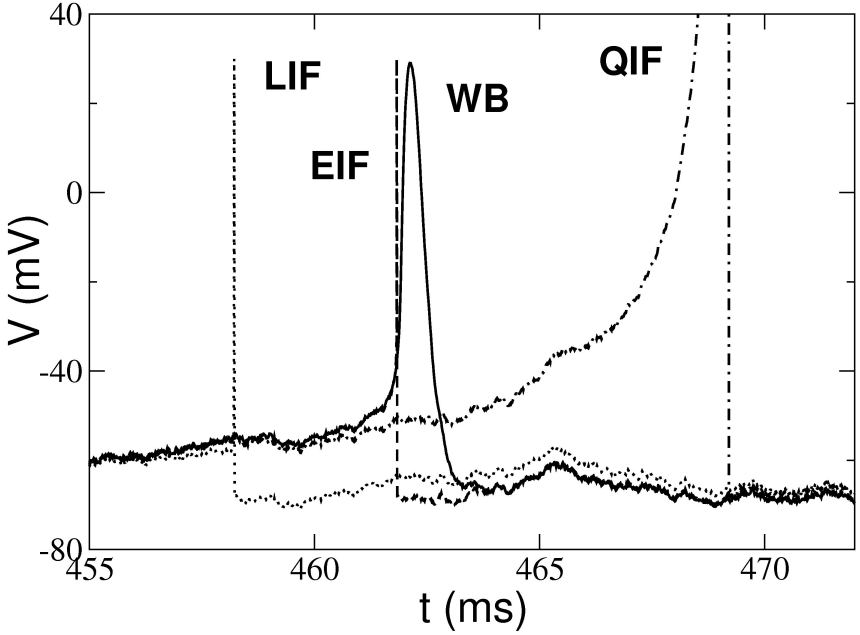
  - **Exponential integrate-and-fire (EIF)**

$$\psi(V) = \Delta_T \exp\left(\frac{V - V_T}{\Delta_T}\right)$$

Nonlinear integrate-and-fire models (cont.)

- EIF describes best spike initiation in HH models

- EIF describes best I-V curves of cortical pyramidal cells and interneurons



Fourcaud-Trocmé et al. 2003; Badel et al. 2008

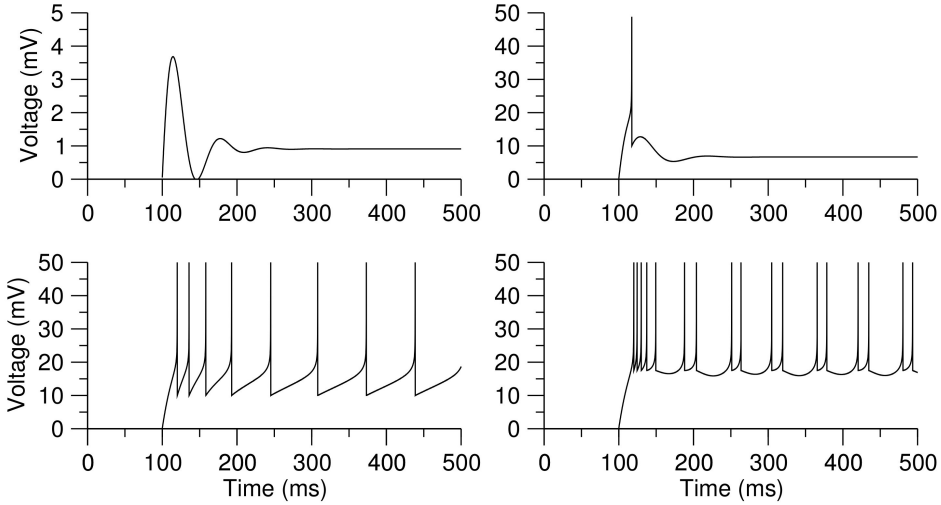
Adaptive integrate-and-fire models

- Add a second variable coupled to the voltage
- Second variable can be driven by voltage itself, by spikes, or by both
- Models in this family:
  - LIF $+$ voltage-driven variable $=$ generalized LIF, resonate-and-fire
  - LIF $+$ spike-driven variable $=$ adaptive LIF
  - QIF $+$ (v&s) driven variable $=$ Izhikevich model
  - EIF $+$ (v&s) variable $=$ refractory EIF (rEIF), adex, adaptive EIF

Richardson et al. 2003; Izhikevich 2003; Brette and Gerstner 2005; Badel et al. 2008

Firing patterns in adaptive integrate-and-fire models

- Adaptive NLIF models can generate a wide diversity of firing patterns:
  - Subthreshold resonance
  - Response with a single spike to constant current injection
  - Firing rate adaptation
  - Bursting
  - Rebound firing
  - etc.



Binary neurons

- Neurons described by binary variables $S_i(t) = 0, 1$ depend on inputs $h_i(t)$ ("local fields"):

$$h_i(t) = I_{iX} + \sum_{j \neq i} J_{ij} S_j(t)$$

- Update rules: specify how S_is are computed from h_is
  - Synchronous updates: define a time step dt and a threshold T
    - Deterministic case:

$$S_i(t + dt) = \Theta(h_i(t) - T) = \begin{cases} 1 & h_i(t) \geq T \\ 0 & h_i(t) < T \end{cases}$$

    - Stochastic case:

$$S_i(t + dt) = \begin{cases} 1 & \text{with probability } \phi(h_i(t)) \\ 0 & \text{with probability } 1 - \phi(h_i(t)) \end{cases}$$

– Asynchronous updates: can happen at any time, with transition rates

$$w(S_i(t) = 0 \to S_i(t) = 1) = \frac{\phi(h_i(t))}{\tau}$$

$$w(S_i(t) = 1 \to S_i(t) = 0) = \frac{1 - \phi(h_i(t))}{\tau}$$

– ϕ sigmoidal function (monotonically increasing from 0 to 1)

$$\phi(x) = \frac{1}{1 + \exp(-\beta(x - T))}$$

where β is analogous to an inverse temperature

Summary - which model neuron to choose?

- To understand biophysical mechanisms: Hodgkin-Huxley type models
- In some conditions, HH type models can be reduced to/approximated by much simpler models
- These simpler models are more amenable to mathematical analysis and computationally cheaper
- LIF-type neurons can reproduce surprisingly well the f-I curve of real neurons, or even the full spike trains of neurons, when simulated by random fluctuating currents
- Binary neurons have been fundamental to understanding the dynamics of various types of networks, such as associative memory models

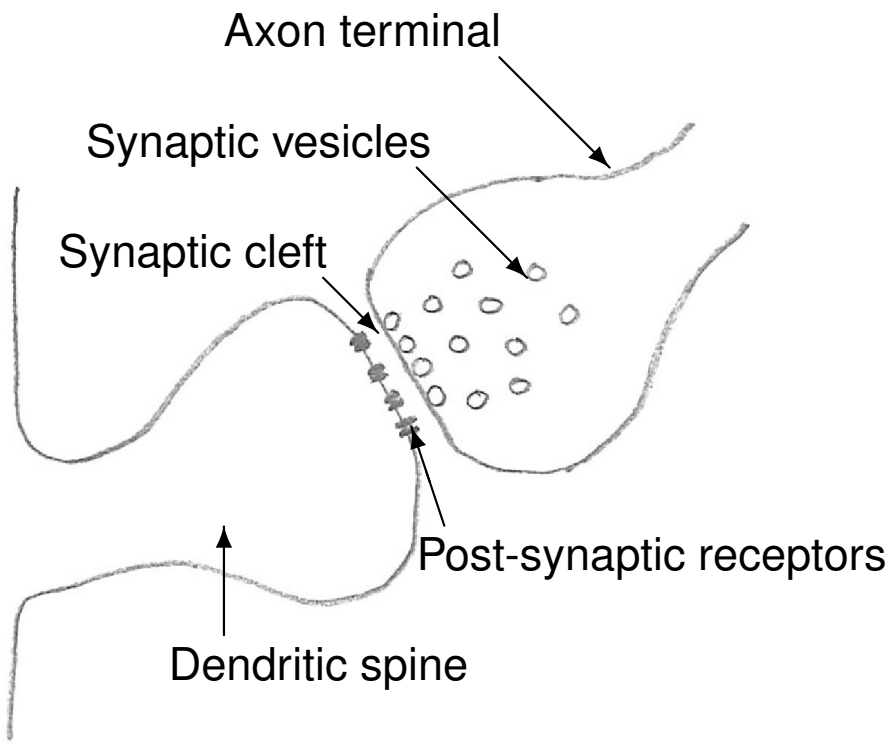
Chapter 2: Basic Neuron and Network Models.

Lecture 4

Synapses

Overview of a synapse

- Action potentials traveling along the axons reach axon terminals (presynaptic boutons)
- Resulting depolarization leads to calcium influx through Ca channels
- Calcium influx causes vesicles to fuse onto the membrane and release neurotransmitters
- Neurotransmitters diffuse in the synaptic cleft and bind with receptor channels located on the membrane of postsynaptic neuron
- Receptor channels open, causing ion influx into the postsynaptic cell



Models of single synapses

- Networks of binary neurons: synapses are described by a single number, the synaptic efficacy
- Networks of spiking neurons: models of synapses are characterized by
  - Time course
  - Voltage dependence
  - History dependence
  - Degree and nature of stochasticity

Time course

- Synaptic currents elicited by spikes of neuron j at times t_j^k can be written as

$$I_j(t) \propto \sum_k S(t - t_j^k)$$

- $S(t)$: time course of an individual postsynaptic current (PSC) triggered by a spike at time $t = 0$
- Popular choices of $S(t)$ in network studies:
  - Delayed delta function

$$S(t) \propto \delta(t - D)$$

  - Instantaneous jump, exponential decay

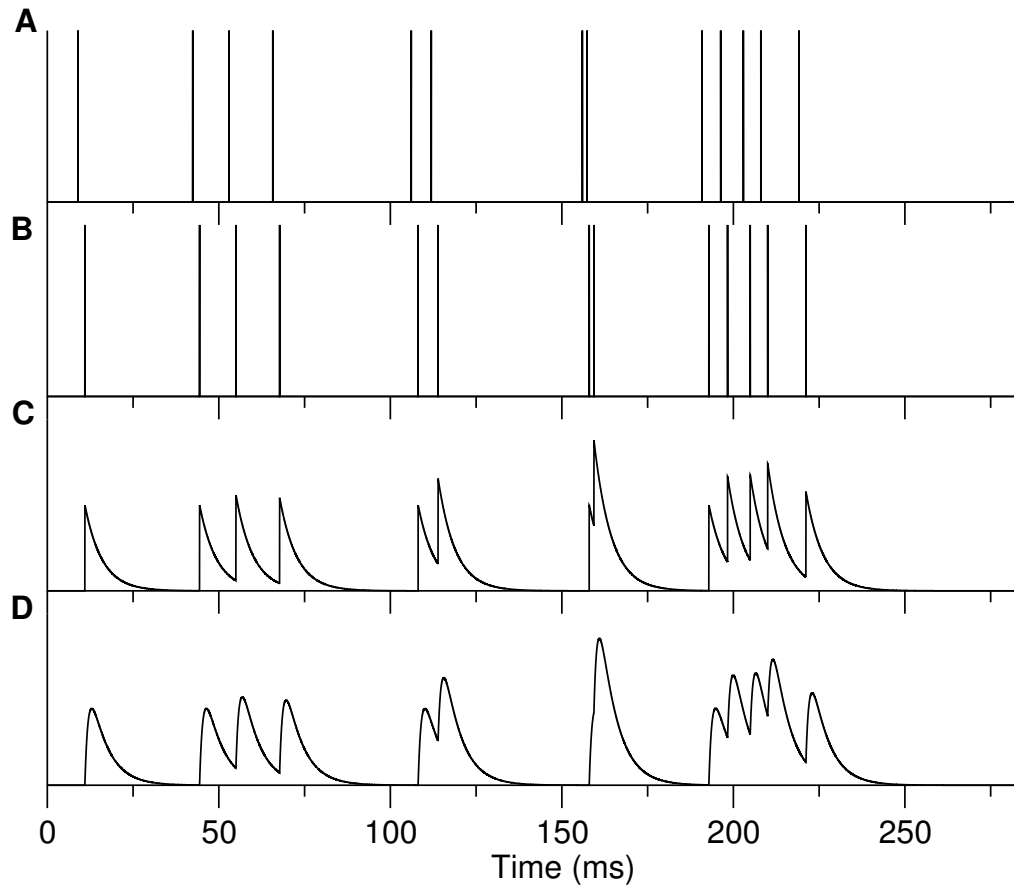
$$S(t) \propto \exp(-t/\tau_s)\Theta(t)$$

  - Difference of exponentials (DOE)

$$S(t) \propto [\exp(-t/\tau_s) - \exp(-t/\tau_r)]\Theta(t)$$

  - Delayed DOE

$$S(t) \propto (\exp(-(t-D)/\tau_s) - \exp(-(t-D)/\tau_r))\Theta(t-D)$$



Voltage dependence

- Current-based models:

$$I_{ij}(t) = J \sum_k S(t - t_j^k)$$

- Conductance-based models:

$$I_{ij}(t) = g(V(t) - V_{syn}) \sum_k S(t - t_j^k)$$

  where

  - $g =$ synaptic conductance
  - $V_{syn} =$ synaptic reversal potential

  Typical values of reversal potential

  - Excitatory synapses: $V_{syn} \sim 0$ mV
  - Inhibitory synapses: $V_{syn} \sim -70$ mV

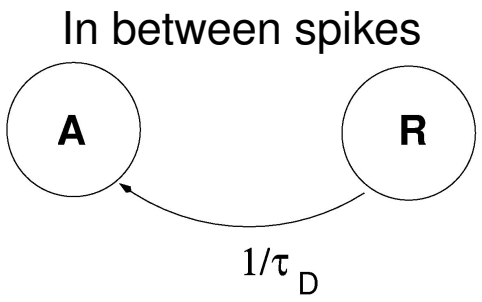
- Voltage dependence of NMDA-mediated currents:

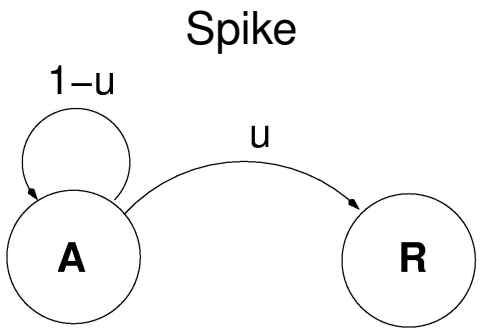
$$g_{NMDA}(V) \sim \frac{1}{1 + \exp(-0.062V)[\mathrm{Mg}^{2+}]/3.57}$$

History dependence

- Synaptic strength is a function of the history of pre- and postsynaptic activity on many different time scales
- Short-term synaptic plasticity describes dynamics on ms-s time scales
  - Short-term depression
  - Short-term facilitation
- Long-term synaptic plasticity describes dynamics on longer time scales (see chapter 6)

Tsodyks-Markram model for short-term depression



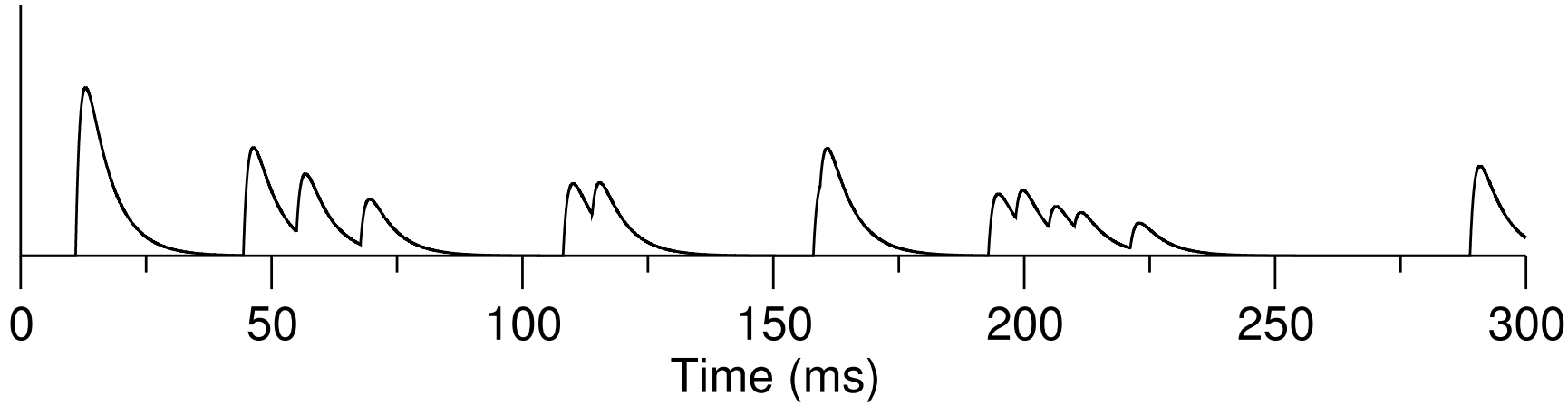


- Model describes two pools of vesicles, those that are available for release (A), and the refractory ones (R)
- Fraction of available vesicles x obey

$$\frac{dx}{dt} = \frac{1-x}{\tau_D} - ux \sum_k \delta(t - t_k)$$

- $u =$ fraction of used vesicles by a presynaptic AP
- $\tau_D =$ recovery time constant
- Postsynaptic current proportional to ux

Tsodyks and Markram 1998



Stochastic nature of synaptic currents

$\Rightarrow$ Quantal model (Katz and collaborators, 1950s):

- N releasable vesicles (release sites)
- p: probability of release of a vesicle (quantum)
- q: postsynaptic response induced by a single released vesicle
- Distribution of responses given by binomial distribution

$$P(I = qn) = C_N^n p^n (1-p)^{N-n}$$

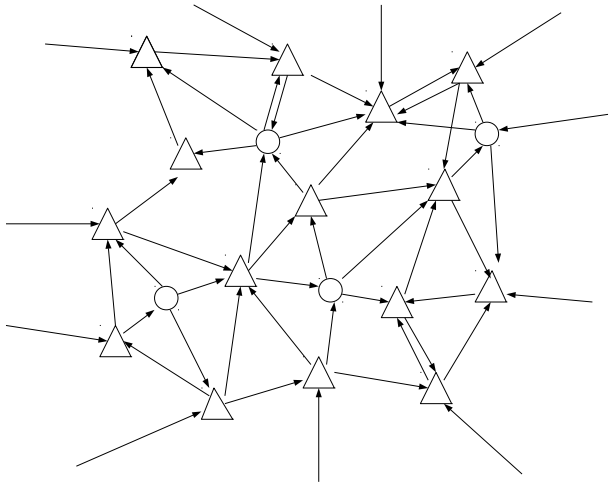
Chapter 2: Basic Neuron and Network Models.

Lecture 5

Networks

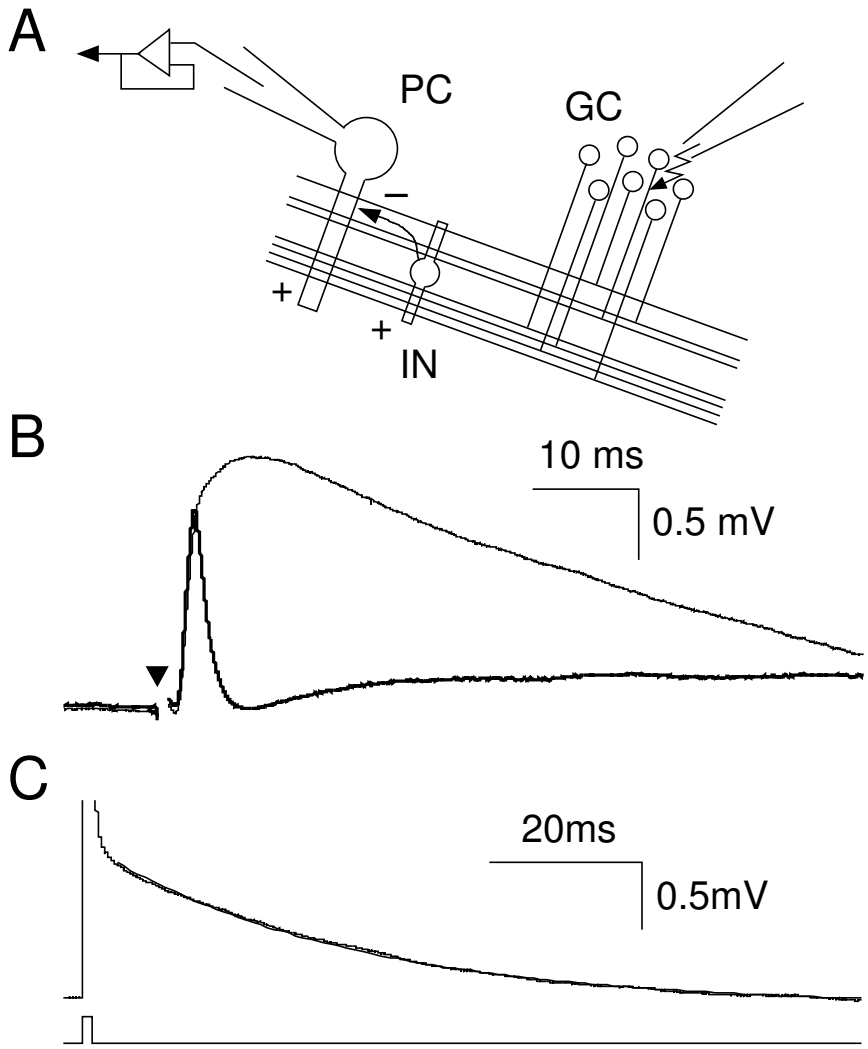
Networks

- Set of N neurons
- Connected through (directed) synaptic connectivity matrix J_{ij} (efficacy of synapse from neuron j to neuron i)
- Some or all neurons may receive external inputs



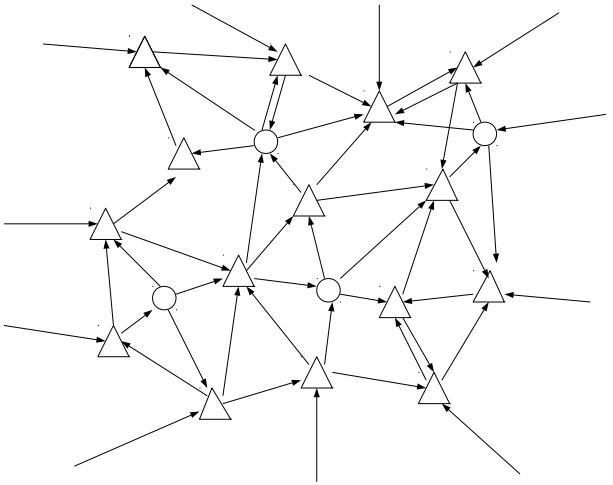
Feedforward networks

- Feedforward networks
  - contain no feedback loops
  - Often arranged as a series of layers, with connections from one layer to the next
  - model networks in sensory systems (early visual system), cerebellar cortex, etc.



Recurrent networks

- Networks with feedback loops
- All neurons potentially connected to all other neurons
- Used to model networks in neocortex and hippocampus in which neurons connect extensively to nearby neurons



Networks of binary neurons

- N neurons
- Neurons described by binary variables $S_i(t) = 0, 1$
- Depend on inputs $h_i(t)$ ("local fields"):

$$h_i(t) = I_{iX} + \sum_{j \neq i} J_{ij} S_j(t)$$

- Update rules: specify how S_is are computed from h_is
  - Synchronous updates: define a time step dt, and a threshold T
    - Deterministic case:

$$S_i(t + dt) = \Theta(h_i(t) - T) = \begin{cases} 1 & h_i(t) \geq T \\ 0 & h_i(t) < T \end{cases}$$

* Stochastic case:

$$S_i(t+dt) = \begin{cases} 1 & \text{with probability } \phi(h_i(t)) \\ 0 & \text{with probability } 1-\phi(h_i(t)) \end{cases}$$

– Asynchronous updates: with transition rates

$$w(S_i(t)=0 \rightarrow S_i(t)=1) = \frac{\phi(h_i(t))}{\tau}$$

$$w(S_i(t)=1 \rightarrow S_i(t)=0) = \frac{1-\phi(h_i(t))}{\tau}$$

– ϕ sigmoidal function (monotonically increasing from 0 to 1)

$$\phi(x) = \frac{1}{1+\exp(-\beta(x-T))}$$

where β is analogous to an inverse temperature

Networks of binary neurons - the symmetric case

- Symmetric network $J_{ij} = J_{ji}$ for all $i \neq j$, with no self-coupling (autapses) $J_{ii} = 0$ for all i
- One can define an **energy function** (or **Lyapunov function**)

$$E(S_1, \ldots, S_N) = -\frac{1}{2} \sum_{j \neq i} J_{ij} S_i S_j - \sum_i (I_{iX} - T) S_i$$

- At zero temperature ($\beta \to \infty$), starting from any initial condition, E decreases monotonically toward a local minimum
- The equilibrium probability of any state $(S_1, \ldots, S_N)$ is given by the **Boltzmann** (or **Gibbs**) **distribution**

$$P(S_1, \ldots, S_N) = \frac{1}{Z} \exp\left(-\beta E(S_1, \ldots, S_N)\right),$$

where $Z = \sum_{S_1, \ldots, S_N} \exp\left(-\beta E(S_1, \ldots, S_N)\right)$ is the **partition function**

The Hopfield (1982) model

- N binary neurons ($S_i(t) = \pm 1$)
- Update rule:

$$S_i(t+1) = \text{sign}\left(\sum_j J_{ij} S_j(t)\right)$$

- p random patterns ξ_i^μ are "memorized" thanks to synaptic matrix

$$J_{ij} = \frac{1}{N}\sum_\mu \xi_i^\mu \xi_j^\mu$$

- Energy function:

$$E = -\frac{1}{2}\sum_{i,j=1}^{N} J_{ij} S_i S_j$$

- "Retrieval states" close to stored patterns if $p < p_{max} \sim 0.14N$
- p_{max} can be computed using methods from statistical physics (Amit, Gutfreund, and Sompolinsky 1985)

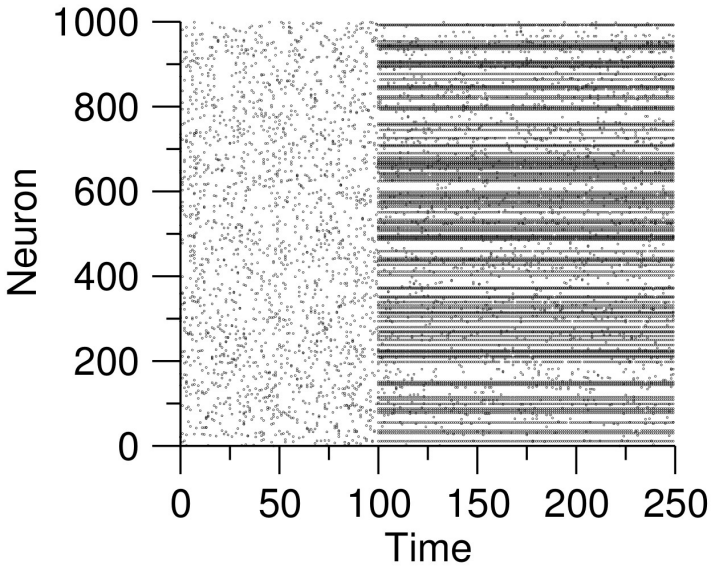
Models with sparse memories

- Generalization to storage of patterns of arbitrary "coding levels": Tsodyks and Feigel'man (1988)
- N binary neurons ($S_i(t) = 0, 1$)
- Neuron dynamics:

$$S_i(t+1) = \Theta\left(\sum_j J_{ij} S_j(t) - T\right)$$

- p "memory states" $\xi_i^\mu = 1$ with probability f (coding level), 0 with probability $1 - f$
- Synaptic matrix $=$ covariance rule

$$J_{ij} = \frac{1}{f(1-f)N} \sum_\mu (\xi_i^\mu - f)(\xi_j^\mu - f)$$



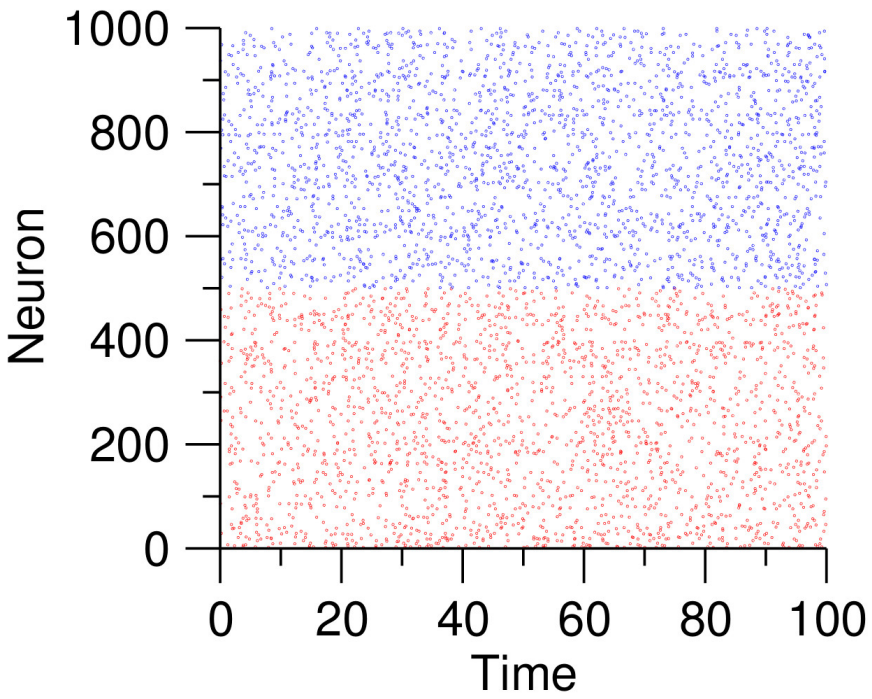
- Selective neurons exhibit *persistent activity* following presentation of one of the stored patterns
- See Fusi and Wang (chapter 11) for a description of more realistic models

Balanced networks

- N_E E neurons, N_I I neurons
- Random sparse connectivity matrix: each neuron receives on average
  - K external E inputs
  - $K \ll N_E$ E recurrent inputs
  - $K \ll N_I$ I inputs
- Strong coupling: coupling strengths $J_{ab} \sim 1/\sqrt{K}$

Irregular firing in balanced networks

- Total excitatory inputs and total inhibitory inputs diverge as $\sqrt{K}$ in the large K limit
- Total inputs stay finite in this limit due to a cancellation of leading order term (*balance between excitation and inhibition*)
- Fluctuations in inputs remain finite in large K limit

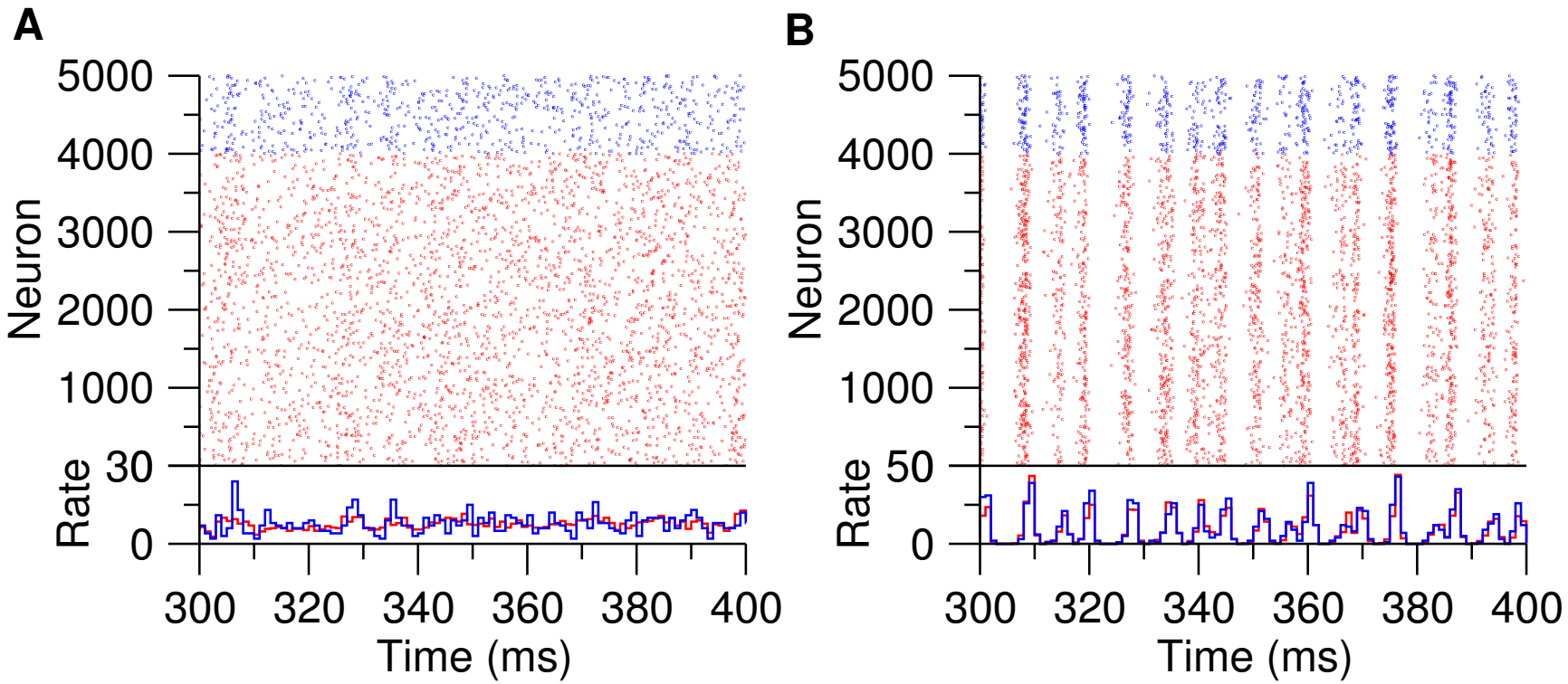


- Neurons fire highly irregularly as a result of large fluctuations in inputs
- Accounts for irregularity of firing of neurons in cortex
- Also leads to broad distributions of firing rates

van Vreeswijk and Sompolinsky 1996, 1998

Networks of spiking neurons - asynchronous and synchronous states

Networks of spiking neurons can settle in asynchronous or synchronous states, depending on parameters



Abbott and van Vreeswijk 1993; Brunel 2000